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Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly From Characteristic Functions

Albert H. Nuttall
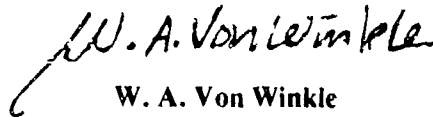Surface Ship Sonar Department

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Newport, Rhode Island / New London, Connecticut

Preface

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An accurate and efficient method of evaluating the entire cumulative or exceedance probability distribution, via one fast Fourier transform of the sampled characteristic function, is presented. The sampling rate applied to the characteristic function results in aliasing of the probability density function, while the limited extent of the sampling gives rise to a systematic disturbance in the calculated probability distribution. Both types of errors are easily recognizable and can be controlled by a trial and error procedure whereby the calculated distributions are plotted for observation and modification.

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The size of the fast Fourier transform determines the number of distribution values available, but has no effect upon the accuracy of the result. Regardless of the number of characteristic function evaluations required for accurate results, the storage required is just that corresponding to the size of the fast Fourier transform.

A program for the procedure is presented, and the inputs required of the user are indicated. Several representative examples and plots illustrate the utility of the approach.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

x	random variable of primary interest
$f_x(\xi)$	characteristic function of random variable x
$p_x(v)$	probability density function of random variable x
b	bias or shift added to x
y	random variable $y=x+b$
$f_y(\xi)$	characteristic function of random variable y
μ_x	mean of random variable x
$p_y(v)$	probability density function of random variable y
$P_x(v)$	cumulative distribution function of random variable x
$P_y(v)$	cumulative distribution function of random variable y
$g(\xi,v)$	auxiliary function (6)
Im	imaginary part
μ_y	mean of random variable y
$C(v)$	right-hand side of (8)
Δ	sampling increment in argument ξ of characteristic function
$\tilde{p}_y(v)$	aliased version of $p_y(v)$; (12)
$\delta_\Delta(\xi)$	impulse train (13)
M	size of FFT employed
z_n	sequence of characteristic function samples; (20)
$\hat{z}_n$	collapsed sequence; (21)
L	limit on integral of characteristic function; (28)
N	number of nonzero z_n; $N=L/\Delta$
overbar	ensemble average

ACCURATE EFFICIENT EVALUATION OF CUMULATIVE OR EXCEEDANCE PROBABILITY DISTRIBUTIONS DIRECTLY FROM CHARACTERISTIC FUNCTIONS

INTRODUCTION

The performance of a signal processor can often be evaluated in terms of the characteristic function of the decision variable, either numerically or in closed form; see for example, refs. 1 and 2. However, a closed form for the corresponding probability density function or cumulative distribution function is seldom available, and numerical procedures must be employed. Several such procedures have been published in the literature, refs. 3-8. However they have limited accuracy or they require extensive storage or analytical manipulations and calculations.

We present a technique which is limited in accuracy only by the round-off noise of the computer or by the errors of the special functions required in the characteristic function calculation. The amount of storage depends only on the number of cumulative or exceedance distribution function values requested and does not influence the accuracy of the final probability values. The entire cumulative and exceedance distribution function values result as the output of one fast Fourier transform (FFT). The size of the FFT dictates the storage required and the spacing of the calculated probability values, but not their accuracy.

The addition and subtraction of integrand functions given in ref. 7 can be entirely circumvented and yet enable use of an FFT, through proper manipulation of the origin contribution of the characteristic function. Specific connections with past results will be noted at appropriate points in the derivations.

DERIVATION OF PROCEDURE

Shifted Random Variable

The primary random variable of interest is the real quantity x with given characteristic function $f_x(\xi)$ which is related to the probability density function p_x of random variable x via Fourier transform *

$$f_x(\xi) = \int dv \ \exp(i\xi v) \ p_x(v). \tag{1}$$

We define secondary random variable y as

$$y = x + b, \tag{2}$$

where bias (shift) b is a constant, chosen such that random variable y has insignificant probability of being less than zero. However, we also pick b as small as possible, so that the characteristic function of y,

$$f_y(\xi) = f_x(\xi) \ \exp(ib\xi), \tag{3}$$

will vary slowly with ξ. In fact, b can be negative, as for example if x were limited to values larger than some positive threshold. The approach here is not limited to positive random variables x, as were some of the results in ref. 7, but is applicable to any random variable distribution.

By way of example, for an exponential probability density function for random variable x, we choose b=0; while for a Gaussian random variable, $b = u_x + 8\sigma_x$ yields a probability less than 1E-15 of y being negative. The probability density function of random variable y therefore appears as depicted in figure 1.

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* Integrals and sums without limits are over $(-\infty, +\infty)$.
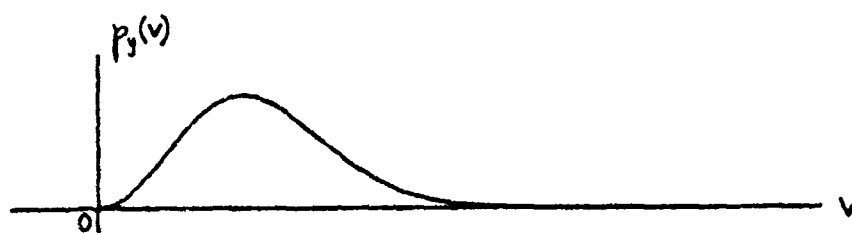
Figure 1. Probability Density Function of Secondary Random Variable y

The cumulative distribution functions of random variables y and x are related according to

$$\int_{-\infty}^{v} dt\, p_y(t) = P_y(v) = P_x(v-b); \qquad P_x(v) = P_y(v+b). \tag{4}$$

Thus we can inspect $P_x(v)$ in the neighborhood of $v=-b$ (the lower edge of interest of x) by looking at cumulative distribution function $P_y(v)$ in the neighborhood of $v=0$. More precisely, we will investigate $P_y(v)$ for values of v greater than zero, since this is the region of significant variation of $P_y(v)$; this is called the positive neighborhood of $v=0$.

Approximation to Cumulative Distribution Function

From ref. 4, eq. 7, we have the cumulative distribution function of random variable y in terms of the characteristic function according to

$$P_y(v) = \frac{1}{2} - \int_{0^+}^{+\infty} d\xi\, g(\xi,v), \tag{5}$$

where we have defined auxiliary function

$$g(\xi,v) = \text{Im}\left\{ \exp(-i\xi v)\, \frac{f_y(\xi)}{\pi\xi} \right\}. \tag{6}$$

Observe for later use that

$$g(0^+,v) = \lim_{\xi \to 0^+} \text{Im}\left\{ (1-i\xi v)\, \frac{1+i\xi u_y}{\pi\xi} \right\} = \frac{u_y - v}{\pi}, \tag{7}$$

where u_y is the mean of random variable y.

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For v in the neighborhood of zero, $\exp(-i\xi v)$ in (6) varies slowly with ξ, and we have the approximation, via the Trapezoidal rule, to (5) as

$$P_y(v) \approx \frac{1}{2} - \frac{\Delta}{2} g(0^+,v) - \sum_{n=1}^{+\infty} \Delta\, g(n\Delta,v) \equiv C(v), \qquad (8)$$

where the right-hand side of (8) has been defined as $C(v)$. Here, Δ is the sampling interval in ξ, and is small enough to track changes in $\exp(-i\xi v) * f_y(\xi)/\xi$. We choose the Trapezoidal rule in (8) over other integration rules, such as Simpson's rule, because it results in minimum aliasing for Fourier transforms relative to all other rules; see appendix A for elaboration and proof.

Observe from (8) that

$$P_y(0) = C(0) \quad \text{means} \quad C(0) = 0, \qquad (9)$$

since $P_y(0)$ is insignificant by the choice of b in (2); this relation will be used later.

Relationship of Approximation

Although we want to evaluate the exact cumulative distribution function $P_y(v)$, we have instead arrived at an approximation $C(v)$ via (8). How are these two related? To determine the relationship, we manipulate (6)-(8) as follows:

$$C(v) = \frac{1}{2} + \frac{\Delta}{2} \frac{v-u_y}{\pi} - \sum_{n=1}^{+\infty} \text{Im}\left\{\exp(-in\Delta v)\, \frac{f_y(n\Delta)}{\pi n}\right\} =$$

$$= \frac{1}{2} + \frac{\Delta}{2} \frac{v-u_y}{\pi} - \text{Im}\left\{\sum_{n=1}^{+\infty} \exp(-in\Delta v)\, \frac{f_y(n\Delta)}{\pi n}\right\} = \qquad (10)$$

$$= \frac{1}{2} + \frac{\Delta}{2} \frac{v-u_y}{\pi} - \frac{1}{i2\pi} \sum_{n\neq 0} \exp(-in\Delta v)\, \frac{f_y(n\Delta)}{n}. \qquad (11)$$

The removal of the imaginary operation from within the summation in (10) is a crucial step; it does not create a problem in divergence since $n > 0$. This is in contrast with the integral of (5) and (6), where removal of the imaginary operation would create a divergent integral. This postponement of the removal of the imaginary operation, until after the approximation to the integral was developed in (8), is the major difference with the results in ref. 7.

Taking a derivative of (11), we obtain

$$C'(v) = \frac{\Delta}{2\pi} + \frac{\Delta}{2\pi} \sum_{n \neq 0} \exp(-in\Delta v) \, f_y(n\Delta) =$$

$$= \frac{\Delta}{2\pi} \sum_n \exp(-in\Delta v) \, f_y(n\Delta) =$$

$$= \frac{1}{2\pi} \int d\xi \, \exp(-i\xi v) \, f_y(\xi) \, \Delta \delta_\Delta(\xi) =$$

$$= p_y(v) \circledast \delta_{\frac{2\pi}{\Delta}}(v) = \sum_n p_y\left(v - n\frac{2\pi}{\Delta}\right) \equiv \tilde{p}_y(v), \tag{12}$$

where infinite impulse train

$$\delta_\Delta(\xi) = \sum_n \delta(\xi - n\Delta), \tag{13}$$

where $\circledast$ denotes convolution, and where we have used the relation

$$\frac{1}{2\pi} \int dt \, \exp(-i\omega t) \, \Delta \delta_\Delta(t) = \delta_{\frac{2\pi}{\Delta}}(\omega). \tag{14}$$

This last result follows from ref. 9, p. 28, rule 11, with $u(t) = \delta(t)$, $T=\Delta$, $F=1/T$, and $\omega=2\pi f$. Relation (12) indicates that $C'(v)$ is an infinitely aliased version of the probability density function $p_y(v)$, with resultant period $2\pi/\Delta$ in v. For small enough sampling increment Δ in (8), there will be very little overlap of the displaced versions of p_y in (12), thereby yielding the good approximation

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$$\tilde{p}_y(v) = p_y(v) \quad \text{for} \quad 0 \leq v \leq 2\pi/\Delta. \tag{15}$$

The situation for relation (12) is depicted in figure 2.
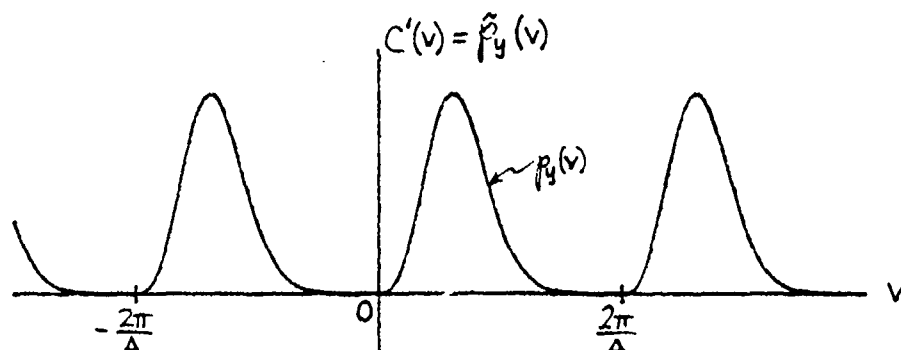
Figure 2. Infinitely Aliased Probability Density Function $\tilde{p}_y(v)$

There now follows from (12),

$$C(v) = C(0) + \int_0^v du \, \tilde{p}_y(u) \equiv C(0) + \tilde{P}_y(v), \tag{16}$$

where $C(0)$ is given by (10) as

$$C(0) = \frac{1}{2} - \frac{\Delta \mu_y}{2\pi} - \text{Im} \left\{ \sum_{n=1}^{+\infty} \frac{f_y(n\Delta)}{\pi n} \right\}. \tag{17}$$

Relation (16) is an exact relation, snowing that $C(v)$ is the integral of the infinitely aliased version of $p_y(v)$, starting at $v=0$, plus an additive constant which is substantially zero; see (9).

So for v in the positive neighborhood of zero, (4), (8), (16), and (9) yield

$$P_x(v-b) = P_y(v) \approx C(v) = C(0) + \tilde{P}_y(v) \approx \tilde{P}_y(v). \tag{18}$$

Thus the quantity we want, the left-most term in (18), is well-approximated by calculated quantity $C(v)$, which itself is approximately the integral of the infinitely aliased version of $p_y(v)$.

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Calculation of C(v)

Let $v = \frac{2\pi k}{M\Delta}$ in (10), where M and k are arbitrary integers. Then

$$C\left(\frac{2\pi k}{M\Delta}\right) = \frac{1}{2} + \frac{k}{M} - \frac{\Delta\mu_y}{2\pi} - \text{Im}\left\{\sum_{n=1}^{+\infty} \exp(-i2\pi nk/M)\frac{f_y(n\Delta)}{\pi n}\right\} =$$

$$= \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi} \text{Im}\left\{\sum_{n=0}^{+\infty} \exp(-i2\pi nk/M)\, z_n\right\}, \tag{19}$$

where we define complex sequence

$$z_n = \begin{cases} i\,\frac{1}{2}\Delta\mu_y & \text{for} \quad n=0 \\ f_y(n\Delta)/n & \text{for} \quad n\geq 1 \end{cases}. \tag{20}$$

Now define collapsed sequence (ref. 7, pp. 13-16) as

$$\hat{z}_n = \sum_{j=0}^{+\infty} z_{n+Mj} \quad \text{for} \quad 0 \leq n \leq M-1. \tag{21}$$

Then since z_n receives the same weight as z_{n+Mj} in (19), regardless of the value of k, (19) can be expressed as

$$C\left(\frac{2\pi k}{M\Delta}\right) = \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi} \text{Im}\left\{\sum_{n=0}^{M-1} \exp(-i2\pi nk/M)\, \hat{z}_n\right\}. \tag{22}$$

Relation (22) is exact and valid for all k. Since we are only interested in the positive neighborhood of $v=0$ in (18), we confine attention in (22) to $0 \leq k \leq M-1$.* Relation (22) can then be accomplished by an M-point FFT if M is chosen to be a power of 2. Notice that storage only for the M complex numbers $\{\hat{z}_n\}$ in (21) is required, even though the $\{z_n\}$ sequence in (20) is of infinite length.

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* Values for other k are available from (22) when we observe that
$C\left(\frac{2\pi(M+k)}{M\Delta}\right) = 1 + C\left(\frac{2\pi k}{M\Delta}\right)$ for all k.

Observe that the size of M in no way affects the error of the calculation of $C(\frac{2\pi k}{M\Delta})$ or estimation of $P_y(v)$. Rather, M specifies the spacing at which $C(\frac{2\pi k}{M\Delta})$ is calculated, and can be coarse if desired. The accuracy of the estimate of $P_y(v)$ is governed thus far by Δ, through the aliasing depicted in figure 2.

Reference to (18) now yields

$$P_x(\frac{2\pi k}{M\Delta} - b) \cong C(\frac{2\pi k}{M\Delta}) \quad for \quad 0 \leq k \leq M-1, \tag{23}$$

where the latter quantity is given by (22). Thus the M-point FFT sweeps out the argument range $(-b, -b+2\pi/\Delta)$ for the cumulative distribution function P_x.

If we want the exceedance distribution function of y instead of the cumulative distribution function, we use (18) and (22) to get

$$1 - C(\frac{2\pi k}{M\Delta}) = \frac{1}{2} - \frac{k}{M} + \frac{1}{\pi} Im\left\{\sum_{n=0}^{M-1} exp(-i2\pi nk/M)\, \hat{z}_n\right\} \quad for \quad 0 \leq k \leq M-1. \tag{24}$$

(By the footnote to (22), we have $1-C(2\pi/\Delta) = -C(0)$.)

Since μ_y must be known in (20) in order to use this approach, we need the mean μ_x of random variable x, since from (2)

$$\mu_y = \mu_x + b. \tag{25}$$

The quantity μ_x can be found analytically from characteristic function $f_x(\xi)$ according to

$$f_x'(0) = i\mu_x; \tag{26}$$

see (1).

In addition to the error caused by aliasing associated with nonzero sampling increment Δ, an additional error occurs because we cannot calculate all the coefficients $\{z_n\}$ in (20) and (21) out to $n=+\infty$. Rather, we terminate the calculation at integer $n=N$, such that $|z_n|$ is sufficiently small as to be negligible for $n \geq N$. Letting

$$L = N\Delta, \tag{27}$$

this is equivalent to ignoring the contribution to (5) of the tail error

$$-\int_L^{+\infty} d\xi \; g(\xi,v) = -\text{Im} \int_L^{+\infty} d\xi \; \exp(-i\xi v) \; \frac{f_y(\xi)}{\pi\xi}. \tag{28}$$

If the asymptotic behavior of $f_y(\xi)$ for large ξ is known, this error can sometimes be evaluated in closed form and used to ascertain an adequate value of L. Instead, we have observed that tail error (28) causes a characteristic low-level sinusoidal variation in the calculated cumulative distribution function for small v near 0, and in the calculated exceedance distribution function for large v near $2\pi/\Delta$. When this sinusoidal variation is deemed excessive, L can be increased until the effect disappears or decreases to acceptable levels. This trial and error approach avoids the necessity of analytically upper bounding the magnitude of error (28), which is often very tedious and generally pessimistic.

So there are two errors to be concerned with: aliasing due to nonzero sampling interval Δ and tail error due to non infinite limit L. Later examples will demonstrate how these errors manifest themselves in the cumulative and exceedance distribution functions and how they can be controlled by a trial and error approach.

Relation to Requicha's Method, ref. 5

From ref. 5, eqs. 7, 9, 10, the cumulative distribution function is given by an expression that can be manipulated into the form (using current notation)

$$F_k = \frac{k}{M} - \frac{1}{\pi} \text{Im} \left\{ \sum_{n=1}^{M/2} \exp(-i2\pi kn/M) \frac{f_y(n\Delta)}{n} \right\} + \frac{1}{\pi} \text{Im} \left\{ \sum_{n=1}^{M/2} \frac{f_y(n\Delta)}{n} \right\}. \quad (29)$$

Although this is similar to the upper line of (19) here, it differs in several important respects:

1. F_k does not use mean μ_y at all; it is therefore not using a direct approximation to the specified integral in (5) and (6).

2. From (29), there follows $F_0 = 0$, $F_M = 1$; however, these results are not strictly true for the actual cumulative distribution function at these end points, thereby leading to poor estimates in the neighborhoods of these points. This is due to the arbitrary origin established in ref. 5, eq. 6.

3. The sums in (29) utilize characteristic function samples $f_y(n\Delta)$ only for $n \leq M/2$, where M is the size of the FFT. This is a very severe and unnecessary restriction; in fact, the sum on n in (29) ought to be conducted to the point where the tail contribution, (28), is negligible, regardless of the value of M.

4. In ref. 5, if eq. 4 is substituted into eq. 1, and the summation limits are extended to $\pm\infty$, we get exactly the second line of (12) here. When the probability density function is integrated to get the cumulative distribution function in ref. 5, eq. 6, the resultant cumulative distribution function is arbitrarily set to zero at v=0. We instead have from (9) and (17),

$$P_y(0) \cong C(0) = \frac{1}{2} - \frac{\Delta\mu_y}{2\pi} - \frac{1}{\pi} \text{Im} \left\{ \sum_{n=1}^{\pm\infty} \frac{f_y(n\Delta)}{n} \right\}, \quad (30)$$

which is small, but not necessarily zero. This consideration is very important on the tails of the cumulative and exceedance distribution functions.

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Summary of Procedure

The cumulative distribution function of y is given by

$$P_y\left(\frac{2\pi k}{M\Delta}\right) \cong C\left(\frac{2\pi k}{M\Delta}\right) = \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi} \, \text{Im}\left\{\sum_{n=0}^{M-1} \exp(-i2\pi nk/M)\, \hat{z}_n\right\}$$

$$\text{for} \quad 0 \leq k \leq M-1, \tag{31}$$

where M is the size of the FFT and storage employed. Also

$$\hat{z}_n = \sum_{j=0}^{+\infty} z_{n+Mj} \quad \text{for} \quad 0 \leq n \leq M-1, \tag{32}$$

where

$$z_n = \begin{cases} i\frac{1}{2}\Delta\mu_y & \text{for} \quad n=0 \\ f_y(n\Delta)/n & \text{for } 1 \leq n \leq N \\ 0 & \text{for } n > N \end{cases}. \tag{33}$$

(The value for n=N should be scaled by 1/2 for the Trapezoidal rule).
The zero values for z_n, when n > N, serve to terminate the collapsed sum in
(32) at a finite upper limit. The value of N is given by the integer part of
L/Δ, where Δ and L must be chosen so as to minimize aliasing and tail error,
respectively. The characteristic function of random variable y needed in (33)
is given by

$$f_y(\zeta) = f_x(\zeta) \exp(ib\zeta), \tag{34}$$

in terms of the characteristic function of the primary random variable x,
where shift b must be chosen such that y = b+x is positive with probabilty
virtually 1. The mean $\mu_y = b+\mu_x$ can be determined analytically from
knowledge of characteristic function $f_x(\zeta)$. Finally, the exceedance
distribution function for random variable y is obtained by subtracting (31)
from 1.

EXAMPLES

Programs for the following five examples are listed in appendix B.

1. Chi-Square

A chi-square variate of 2K degrees of freedom has probability density function (ref. 10)

$$p_x(v) = \frac{v^{K-1} \exp(-v/2)}{2^K(K-1)!} \quad \text{for } v > 0 \tag{35}$$

and characteristic function

$$f_x(\xi) = (1-i2\xi)^{-K}. \tag{36}$$

Since random variable x is obviously nonnegative by (35), we can choose shift b=0; i.e. y=x. A plot of the cumulative and exceedance distribution functions of random variable y obtained from characteristic function (36) with K=4 is given in figure 3 for $0 \le v \le 2\pi/\Delta$. The values of Δ and L have been chosen such that aliasing and tail error are insignificant.

The ordinate scale for figure 3 is a logarithmic one. The lower right end of the exceedance distribution function curve decreases smoothly to the region 1E-11, where round-off noise is encountered. The exceedance distribution function values continue to decrease with v until, finally, negative values (due to round-off noise) are generated. For negative probability values, the logarithm of the absolute value is plotted, but mirrored below the 1E-12 level. These values have no physical significance, of course; they are plotted to illustrate the level of accuracy attainable by this procedure with appropriate choices of Δ and L.

For this example, $N=L/\Delta=2666$, while M=256. Thus collapsing, according to (21) or (32), by over a factor of 10 has been employed and a small size FFT has been utilized. Nevertheless the error realized for the cumulative and exceedance distribution functions is in the 1E-12 range, the limit of accuracy of the Hewlett Packard 9845B Desk Calculator used here. Finer spacing in the distribution outputs is achievable by merely increasing M.
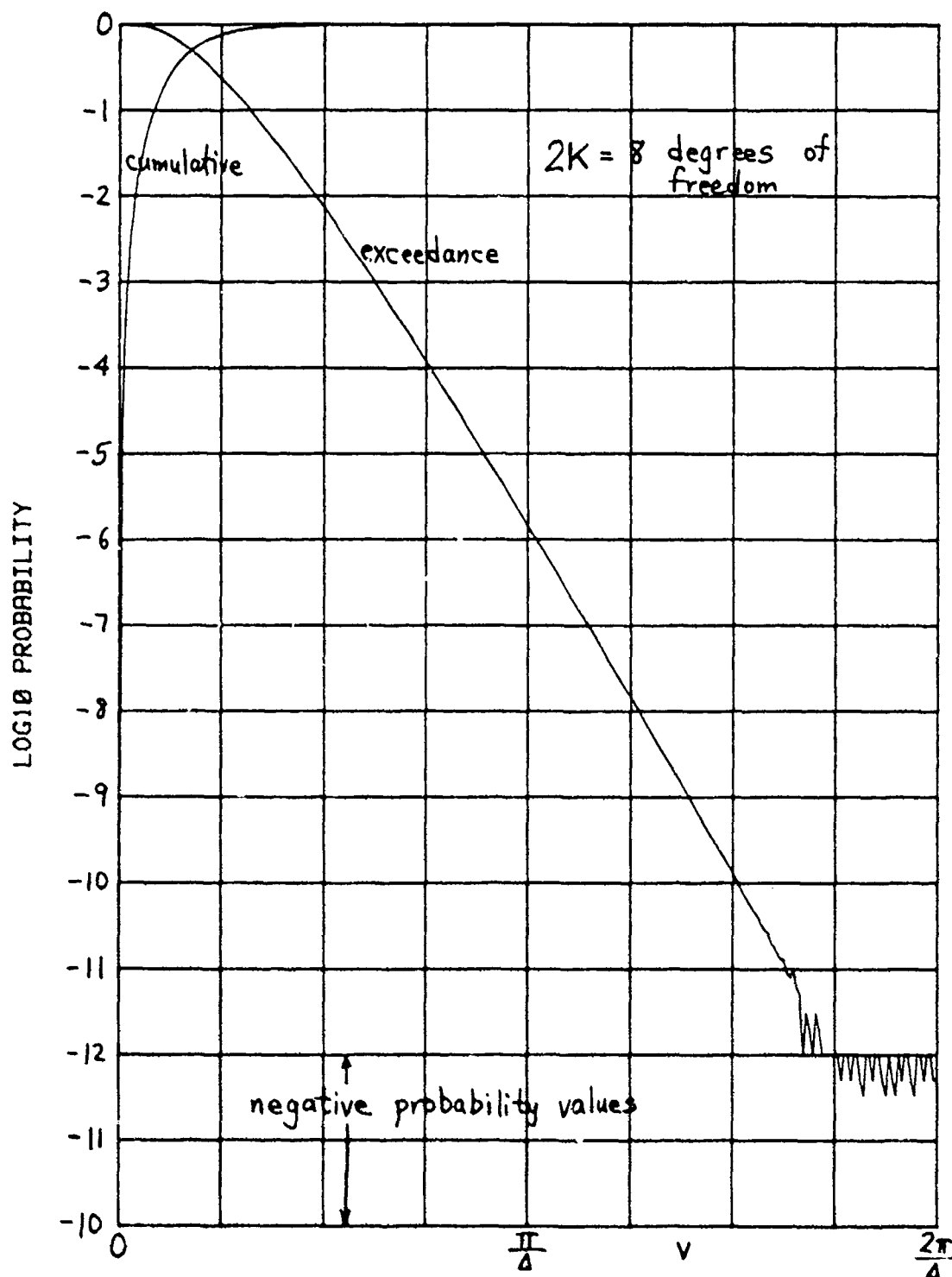
Figure 3. Chi-Square; L=200, Δ=.075, b=0, M=256

2. Gaussian

The characteristic function for a zero-mean unit-variance random variable is

$$f_x(\xi) = \exp(-\xi^2/2), \tag{37}$$

and the probability density function and cumulative distribution function are (ref. 11, eq. 10.5.3)

$$p_x(v) = (2\pi)^{-1/2} \exp(-v^2/2), \quad P_x(v) = \Phi(v). \tag{38}$$

For $b = 5\pi/2$, using (4),

$$P_y(0) = P_x(-b) = \Phi(-b) = 2E\text{-}15. \tag{39}$$

which is negligible, as desired.

Plots of the cumulative and exceedance distribution functions for random variable y are given in figure 4 for $L=7$, $\Delta=.3$. The logarithmic ordinate gives rise to the characteristic parabolic shape on the tails of the distributions. Once again, the probabilities decrease to the level of the round-off noise and fluctuate around 1E-12 near the edges of the fundamental aliased interval $(0, 2\pi/\Delta)$. The fact that the cumulative distribution function of y starts in the round-off noise at $v=0$ indicates that $b=5\pi/2$ was large enough to guarantee $y > 0$ with probability virtually 1. Also indicated on the figure is the origin for random variable x. We have, from (4),

$$P_x(u) = P_y(u+b); \tag{40}$$

thus for example

$$\text{Prob}(x < 0) = P_x(0) = P_y(b) = .5. \tag{41}$$

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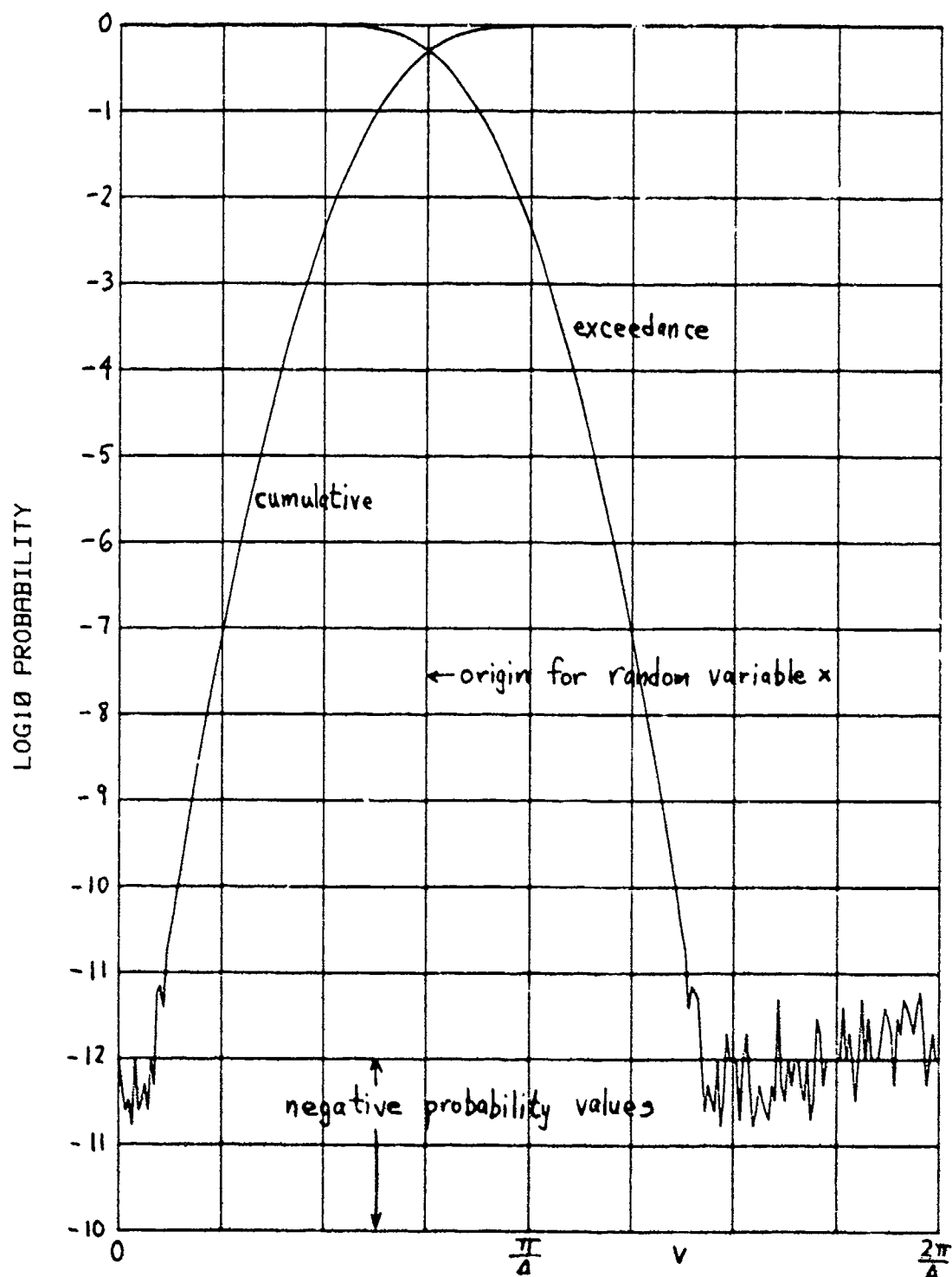
Figure 4. Gaussian; L=7, Δ=.3, b=2.5π, M=256

In figure 5, the only change is to decrease limit L from 7 to 6. The tail error mentioned in (28) et seq. then dominates the round-off noise and has a sinusoidal variation. Aliasing is not a problem, as witnessed by the fact that the cumulative and exceedance distribution functions of random variable y have decayed below 1E-12 well before the edges of the interval are reached.

When limit L is restored to 7, and sampling increment Δ is increased to .5, aliasing becomes significant, as shown in figure 6. The exceedance distribution function has not yet decayed to the round-off noise level at v=2π/Δ, and the cumulative distribution function shows a large negative probability region near v=0. Shift b has been maintained at the value 5π/2, corresponding to (39).

When L and Δ are restored to their values 7 and .3 as for figure 4, but b is decreased to 5π/3, the probability of y becoming negative is, from (4) and (38), $\Phi(-5\pi/3)$ = .82E-7. This is reflected in the cumulative distribution function for y in figure 7 at v=0, where the probability value is well above the round-off noise level. Also, the exceedance distribution function develops significantly negative values near v = 2π/Δ.

Accurate evaluation of the cumulative and exceedance distribution functions can only be achieved when L, Δ, and b are properly chosen. Probably the optimum combination for the Gaussian variate is displayed in figure 8, where Δ has been increased to .4, the distributions are centered on the fundamental aliased interval (0, 2π/Δ) by choice of b, and L is taken at 7 to avoid tail error.

3. Smirnov

The limiting characteristic function of a measure of goodness of fit based on the sample distribution function was derived by Smirnov and is given by (ref. 12, eq. 30.104)

$$f_\lambda(\zeta) = \left(\frac{s}{\sin(s)}\right)^{1/2} \quad \text{where } s = (1+i)\sqrt{\zeta} \quad \text{for } \zeta \geq 0. \tag{42}$$
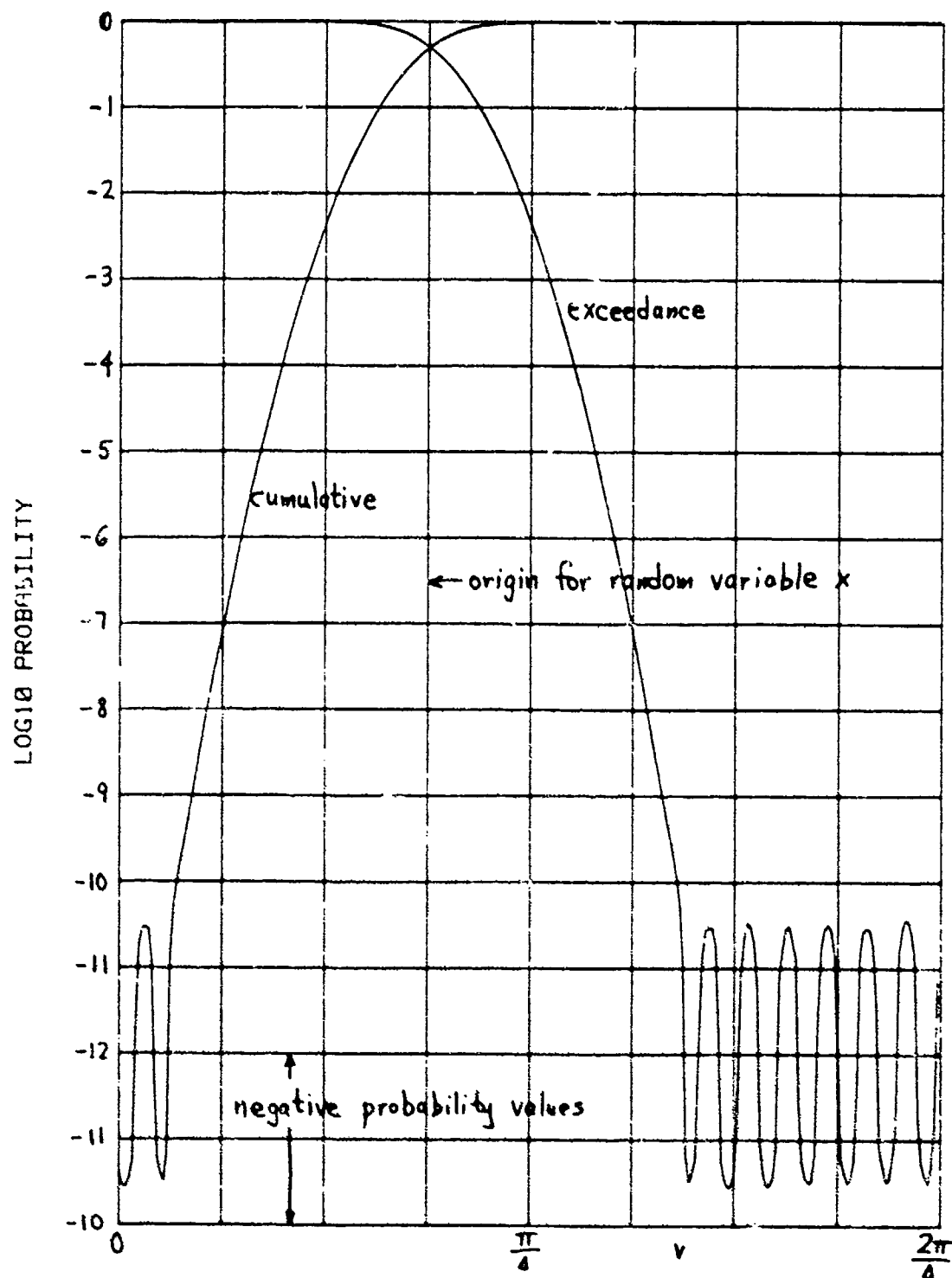
Figure 5. Gaussian; L=6, Δ=.3, b=2.5π, M=256
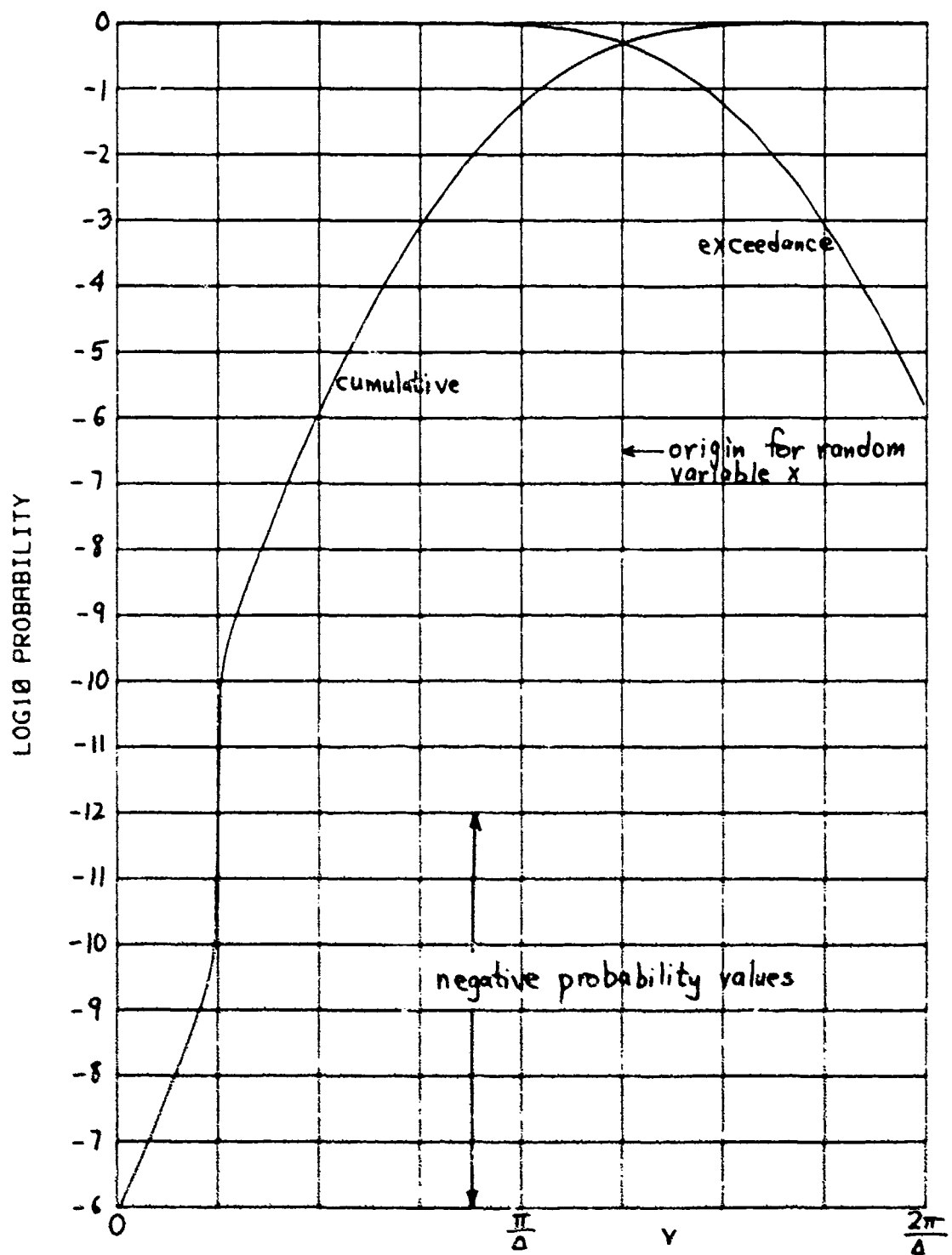
Figure 6. Gaussian; L=7, Δ=.5, b=2.5π, M=256
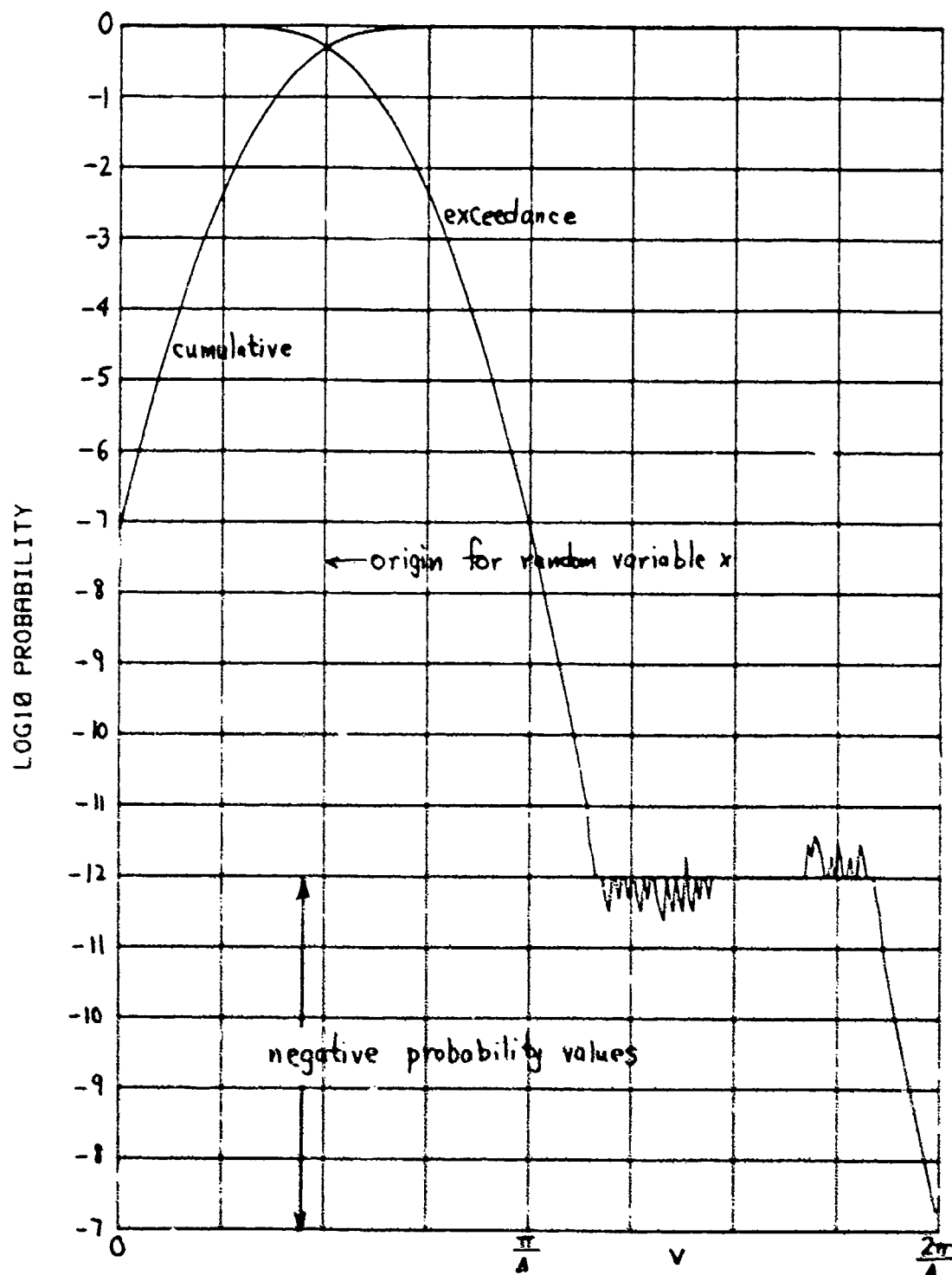
Figure 7. Gaussian; L=7, Δ=.3, b=5π/3, M=256
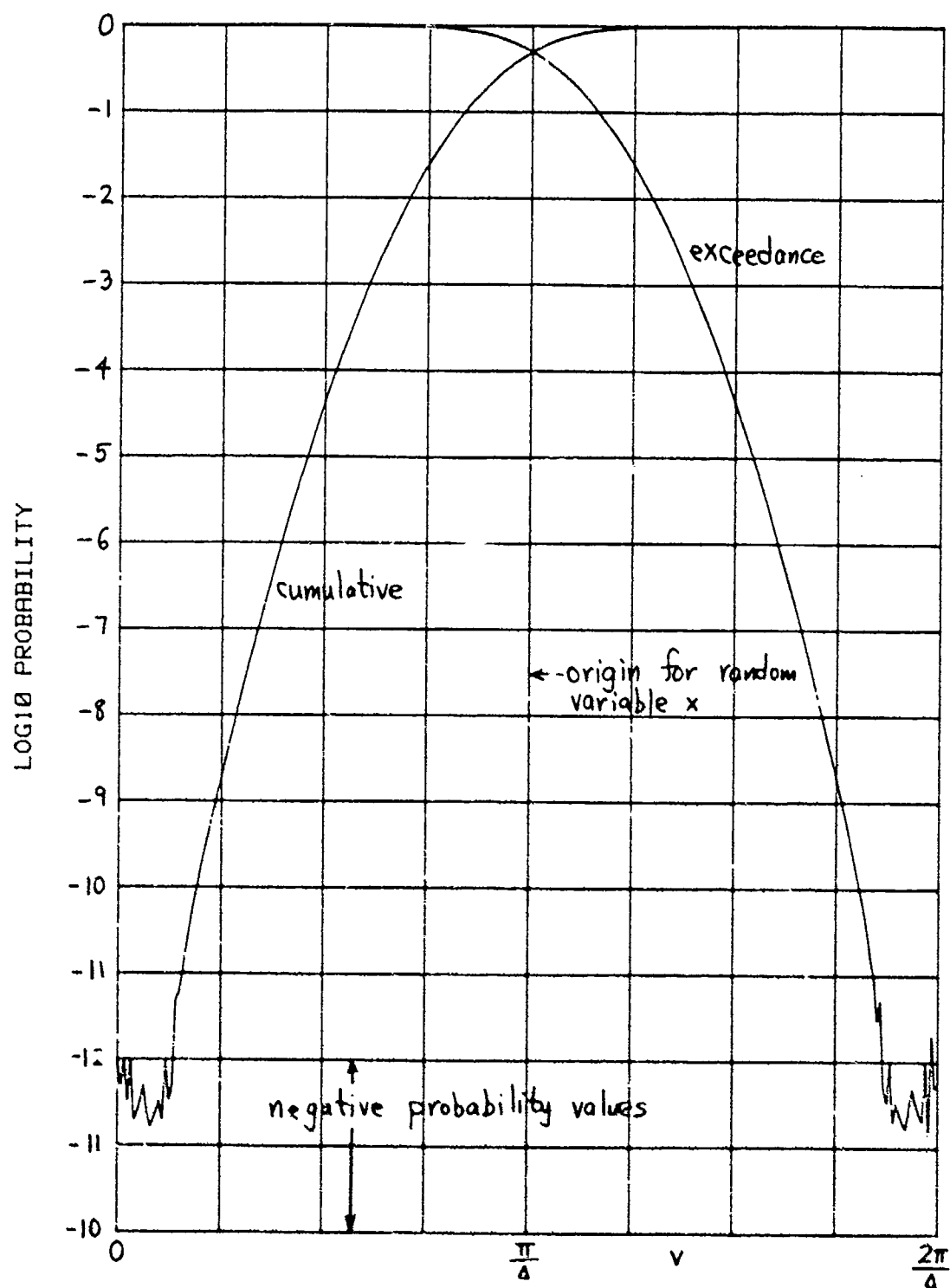
Figure 8. Gaussian; L=7, Δ=.4, b=2.5π, M=256

An expansion about $\zeta=0$ yields

$$f_x(\zeta) = 1 + i\frac{1}{6}\zeta - \frac{1}{40}\zeta^2; \quad i.e., \mu_x = 1/6, \sigma_x^2 = 1/45. \tag{43}$$

And since the goodness of fit is always positive, random variable x is positive and we can choose

$$b=0. \tag{44}$$

Since

$$\sin((1+i)\sqrt{\zeta}) \sim i\frac{1}{2}\exp(\sqrt{\zeta}(1-i)) \quad \text{as } \zeta\to+\infty, \tag{45}$$

it follows that

$$f_x(\zeta) \sim 2^{3/4}\zeta^{1/4}\exp(-\frac{1}{2}\sqrt{\zeta} + i(\frac{1}{2}\sqrt{\zeta} - \frac{\pi}{8})) \quad \text{as } \zeta\to+\infty. \tag{46}$$

The phase of this term rotates according to $\sqrt{\zeta}/2$; if we were to choose $b\neq0$, $f_y(\zeta)$ would rotate faster than $f_x(\zeta)$ (linear with ζ rather than $\sqrt{\zeta}$). This could necessitate a faster sampling rate, which is undesirable.

The cumulative and exceedance distribution functions are plotted in figure 9. L and Δ have been chosen so as to avoid tail error and aliasing. The exceedance distribution function is seen to decay exponentially until it reaches approximately 2E-11; the bump in the curve at this point is a manifestation of the limited accuracy of the trigonometric functions built into the calculator employed. Larger values of v lead to round-off noise around the 1E-12 level.

A comparison of results for this characteristic function, with Requicha's method described in (29) et seq., is given in figure 10 for FFT size M=1024. The plot labeled with N=L=512 is precisely Requicha's method. Aliasing is known to be insignificant for $\Delta=1$, as seen by reference to figure 9 and observing that extrapolation of the straight line section of the exceedance distribution function would result in probability values near 1E-13 at $v=2\pi/\Delta$. The dashed portion of the N=L=512 curve in figure 10 in fact

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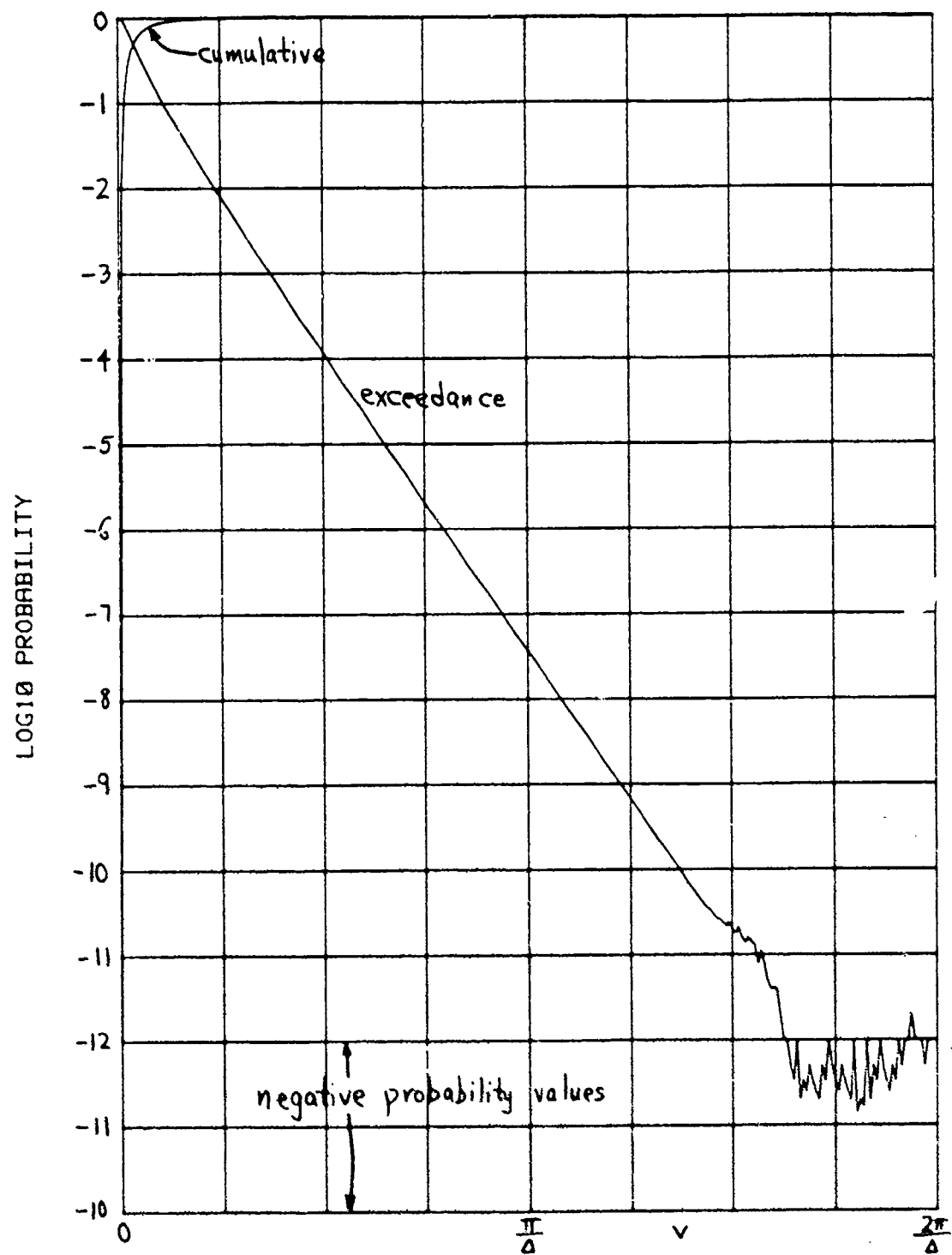
Figure 9. Smirnov; L=3000, Δ=1, b=0, M=256
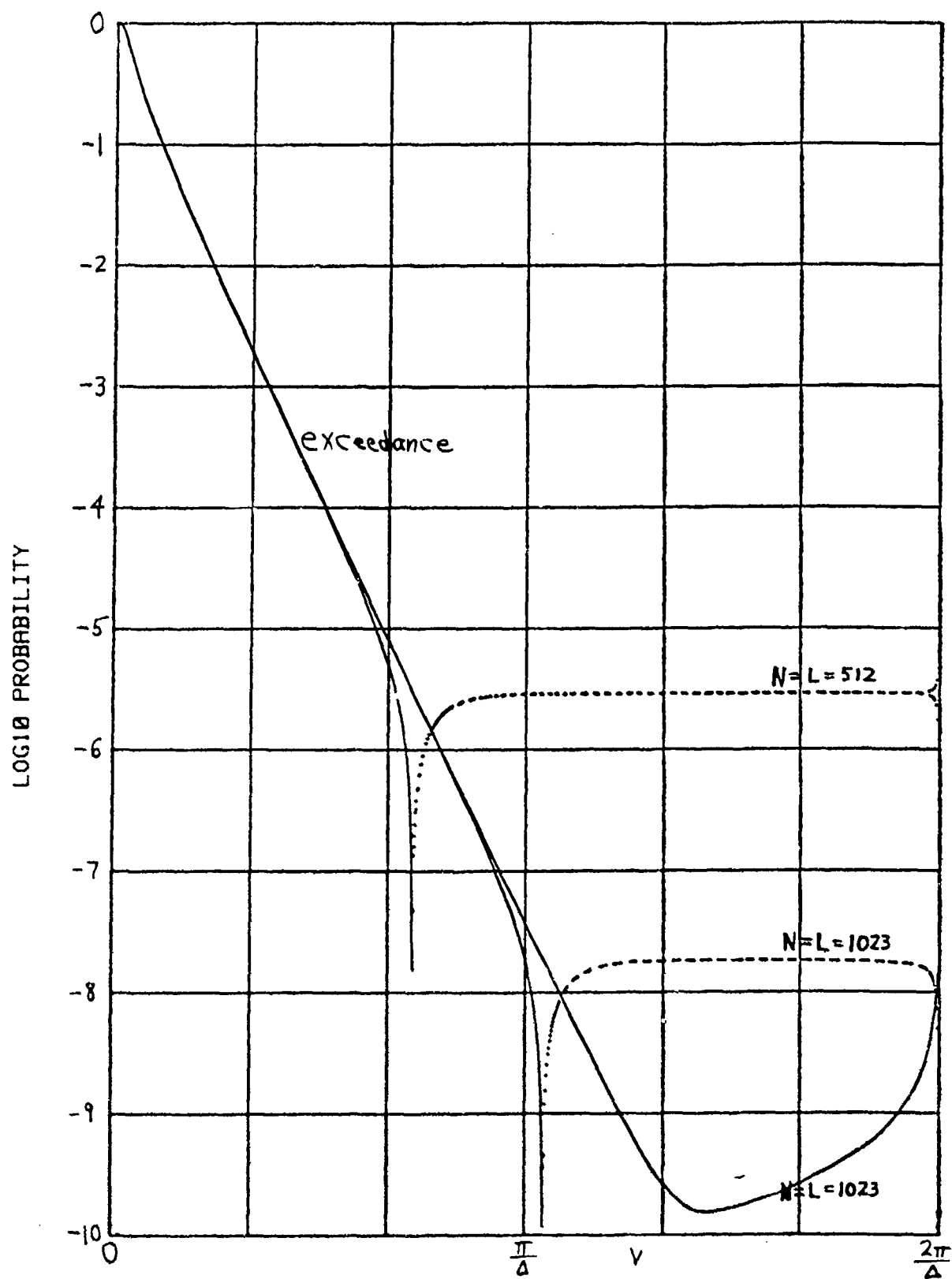
Figure 10. Comparison with Requicha's Method;
$\Delta = 1$, $b = 0$, $M = 1024$

corresponds to negative probability estimates; these grossly inaccurate results are due to an inadequate value of limit L, leading to large tail error.

When N is simply increased to 1023, the middle curve in figure 10 results from Requicha's method. Again, negative estimates are indicated by the dashed portion of this curve, although two orders of magnitude smaller than above. The reasons for these errors have been delineated in (29) et seq.

The bottom-most curve in figure 10 (solid curve) is that obtained by the method proposed in this report for L = 1023. Exceedance distribution function estimates in the 1E-10 range are obtained, but the error returns to the 1E-8 range at $v=2\pi/\Delta$. No negative probability values occur. Also, by simply increasing limit L, while keeping FFT size M fixed, the error can be reduced significantly further, as already witnessed by figure 9.

4. Noncentral Chi-Square

Here the random variable x is given by

$$x = \sum_{k=1}^{K} (g_k + d_k)^2 \tag{47}$$

where $\{d_k\}$ are constants, and $\{g_k\}$ are independent Gaussian random variables with zero-mean and unit variance. The characteristic function of x is

$$f_x(\zeta) = (1-i2\zeta)^{-K/2} \exp\left(\frac{id^2\zeta}{1-i2\zeta}\right) , \tag{48}$$

where deflection d is defined according to

$$d^2 = \sum_{k=1}^{K} d_k^2. \tag{49}$$

We actually consider a more general characteristic function than (48), namely

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$$f_X(\zeta) = (1-i2\zeta)^{-\nu} \exp\!\left(\frac{id^2\zeta}{1-i2\zeta}\right) = \exp\!\left(\frac{id^2\zeta}{1-i2\zeta} - \nu \ln(1-i2\zeta)\right), \qquad (50)$$

where ν is an arbitrary positive real constant. Suppose that we use the principal value logarithm for $\ln(z)$, where the branch cut lies along the negative real axis of the complex z plane (ref. 13, sect. 4.1.1). Then since the argument of the logarithm in (50) never crosses the branch cut, form (50) gives the correct characteristic function values automatically for all real ζ, and any ν.

The probability density function and exceedance distribution function corresponding to (50) are (ref. 14, 6.631 4)

$$p_X(v) = \frac{1}{2} \exp\!\left(-\frac{d^2+v}{2}\right)\!\left(\frac{\sqrt{v}}{d}\right)^{\nu-1} I_{\nu-1}(d\sqrt{v}) \quad \text{for} \quad v > 0,$$

$$1 - P_X(v) = \int_{\sqrt{v}}^{+\infty} dt \; t \exp\!\left(-\frac{d^2+t^2}{2}\right)\!\left(\frac{t}{d}\right)^{\nu-1} I_{\nu-1}(dt) \equiv Q_\nu(d,\sqrt{v}) \quad \text{for} \quad v > 0. \quad (51)$$

Since the probability density function in (51) is never negative (ref. 13, sect. 9.6.1), (50) is a legal characteristic function. Also because random variable x is always positive according to (51), we choose shift b=0. Plots of the exceedance distribution function, as determined from characteristic function (50) are displayed for various values of d in figure 11. The values of L were chosen for each d value so as to control the tail error below the 1E-10 level plotted. Direct calculation of the exceedance distribution function directly from (51) would be a formidable task for arbitrary ν values.

5. Product of Correlated Gaussian Variates

Let

$$x = st \qquad (52)$$

where s and t are zero-mean unit-variance Gaussian random variables with correlation coefficient ρ. The joint probability density function of s and t is

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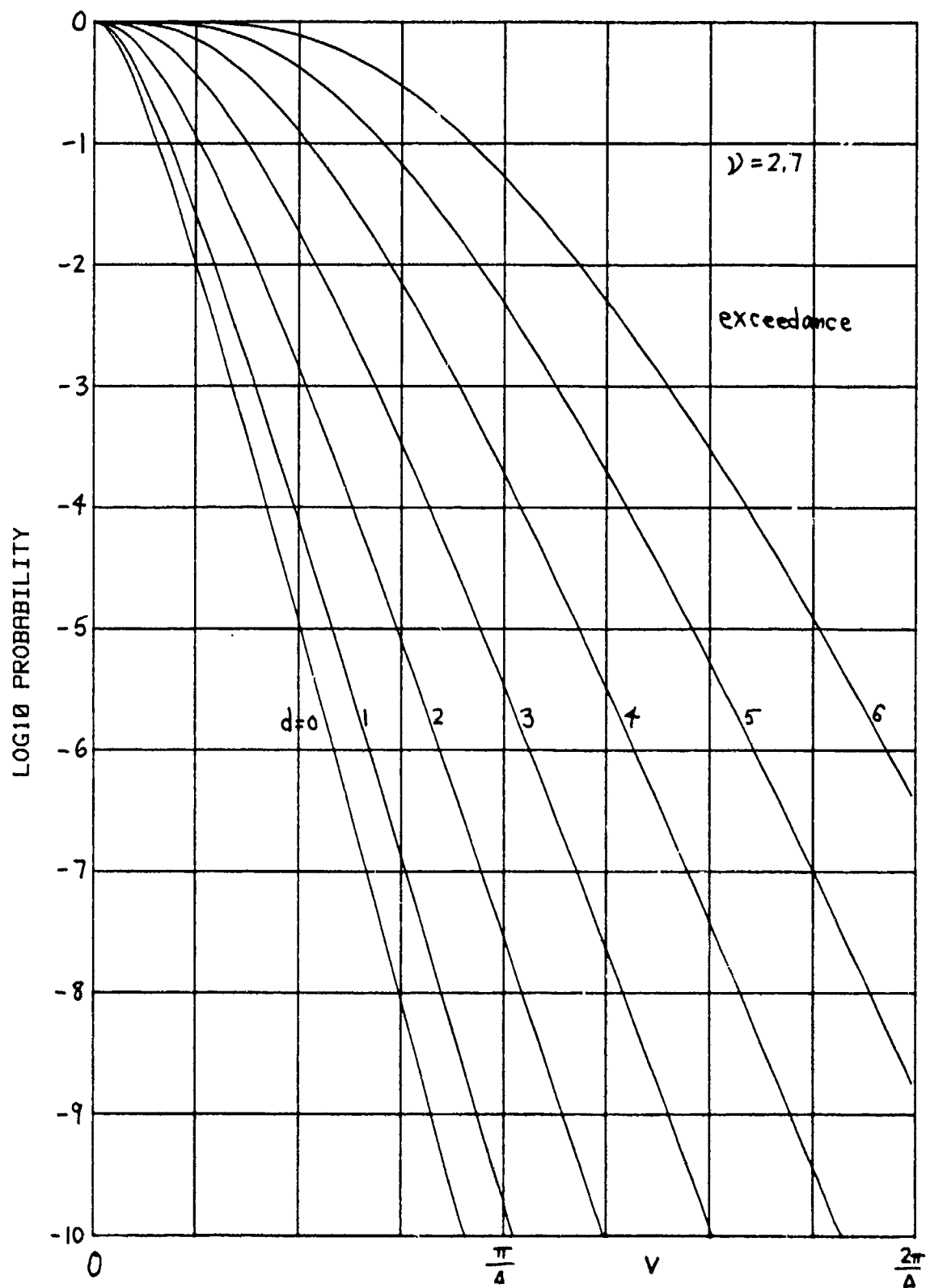
Figure 11. Non-Central Chi-Square; $\Delta = .05$, $b = 0$, $M = 256$

$$p_{st}(u,v) = \left(2\pi\sqrt{1-\rho^2}\right)^{-1} \exp\left[-\frac{u^2+v^2-2\rho uv}{2(1-\rho^2)}\right]. \qquad (53)$$

The characteristic function of x is then

$$f_x(\xi) = \overline{\exp(i\xi st)} = \iint du\,dv\,\exp(i\xi uv)\,p_{st}(u,v) =$$

$$= \left[1-i2\rho\xi+(1-\rho^2)\xi^2\right]^{-1/2} = \left[1-i(1+\rho)\xi\right]^{-1/2}\left[1+i(1-\rho)\xi\right]^{-1/2}, \qquad (54)$$

via repeated use of ref. 14, eq. 3.323 2. The corresponding probability density function of x is

$$p_x(v) = \int\frac{dy}{|y|}\,p_{st}\left(y,\frac{v}{y}\right) = \frac{1}{\pi\sqrt{1-\rho^2}}\exp\left(\frac{\rho v}{1-\rho^2}\right)K_0\left(\frac{|v|}{1-\rho^2}\right) \text{ for all } v, \qquad (55)$$

via ref. 14, eq. 3.478 4.

(If we transform this probability density function according to (1) and use ref. 14, eq. 6.611 9 and ref. 13, eq. 4.4.15, we get precisely (54). Alternatively, if we transform (54) and modify the contour to wrap around the branch line along the imaginary axis and then use ref. 14, eq. 3.388 2, we get (55). Or we can use ref. 14, eq. 3.754 2.)

We actually consider a more general characteristic function than (54), namely

$$f_x(\xi) = \left[1-i2\rho\xi+(1-\rho^2)\xi^2\right]^{-\nu} = \exp\left(-\nu\ln\left[1-i2\rho\xi+(1-\rho^2)\xi^2\right]\right) =$$

$$= \left[\frac{1}{1-\rho^2}+\left\{\sqrt{1-\rho^2}\,\xi-i\frac{\rho}{\sqrt{1-\rho^2}}\right\}^2\right]^{-\nu}. \qquad (56)$$

The mean of this random variable x is given by

$$\mu_x = 2\nu\rho. \qquad (57)$$

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The probability density function corresponding to (56) is

$$p_X(v) = \frac{1}{2\pi} \int d\xi \; \exp(-i\xi v) \; f_X(\xi) =$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int\limits_{-\infty-i\rho/\sqrt{1-\rho^2}}^{+\infty-i\rho/\sqrt{1-\rho^2}} dy \; \exp\left(\frac{\rho v}{1-\rho^2} - i\frac{yv}{\sqrt{1-\rho^2}}\right)\left(\frac{1}{1-\rho^2} + y^2\right)^{-\upsilon}, \quad (58)$$

where we let

$$y = \sqrt{1-\rho^2}\,\xi - i\frac{\rho}{\sqrt{1-\rho^2}}. \qquad (59)$$

We can move the contour in (58) to the real y-axis, because the branch points of the integrand are at $y = \pm\, i/\sqrt{1-\rho^2}$ which are outside the path of integration, since $|\rho| < 1$. Then using ref. 14, eq. 3.771 2 and ref. 13, eq. 6.1.17, we obtain

$$p_X(v) = \left(\sqrt{\pi}\;\Gamma(\upsilon)\sqrt{1-\rho^2}\right)^{-1}\left(\frac{|v|}{2}\right)^{\upsilon-\frac{1}{2}}\exp\left(\frac{\rho v}{1-\rho^2}\right)K_{\upsilon-\frac{1}{2}}\left(\frac{|v|}{1-\rho^2}\right) \quad \text{for all v.} \quad (60)$$

Since this probability density function is never negative (ref. 13, sect. 9.6.1), (56) is a legal characteristic function. If we Fourier transform (60) via ref. 14, 6.699 12, we get (56) directly.

There is no simple relation for the cumulative distribution function of this random variable. Nevertheless, it is a simple matter to evaluate directly from characteristic function (56). The $\ln$ in (56) causes no problems since its argument never crosses the branch cut. A plot for $\upsilon=7.7$ and $\rho=-.3$ is displayed in figure 12. The rate of decay of the distribution is different for each tail. The round-off noise is clearly visible at both ends of the range of v values.
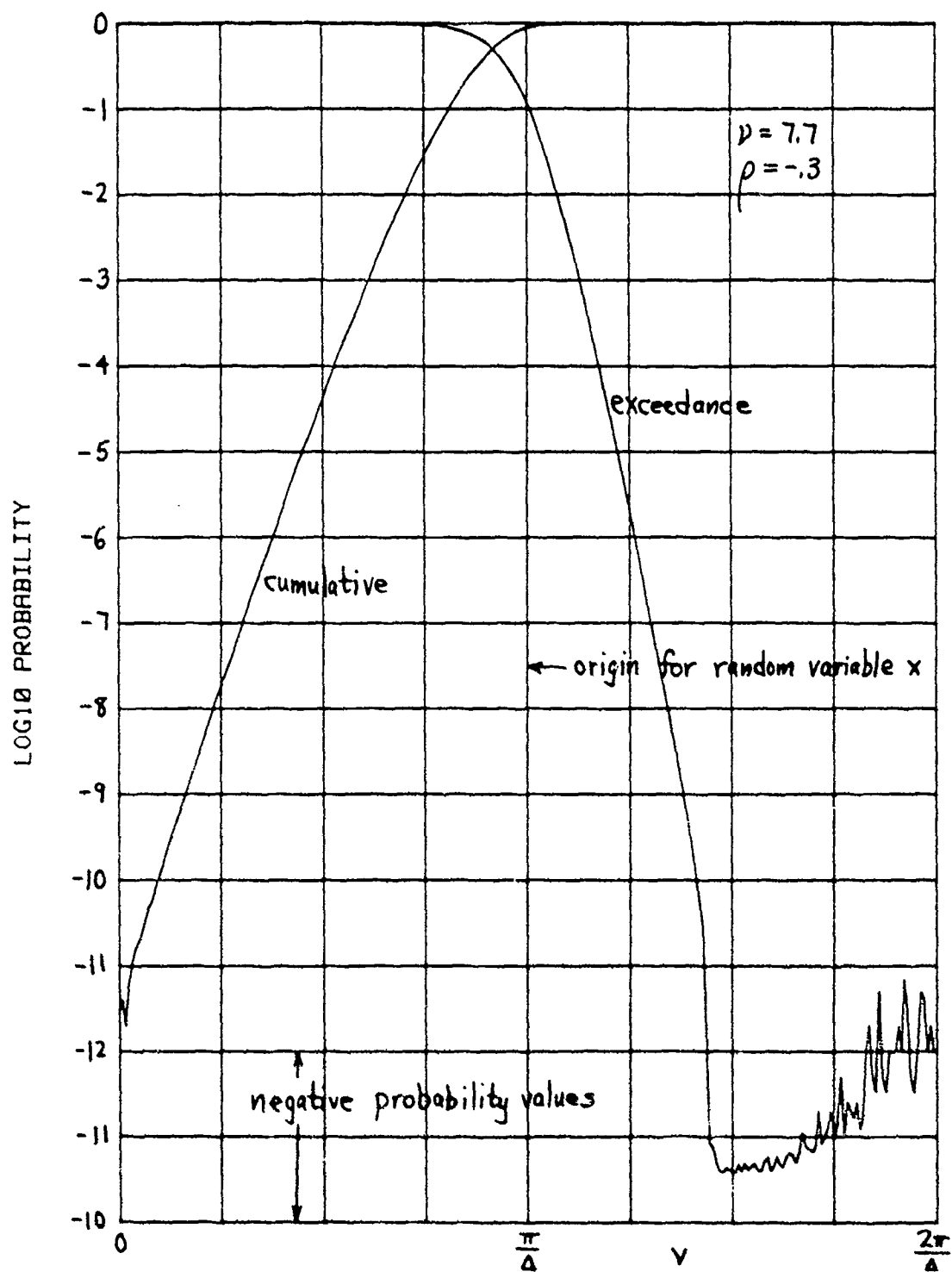
Figure 12. Gaussian Product; L=5, Δ=.06, b=50π/3, M=256

APPLICATIONS

We now have the capability to handle the following type of statistical problem in a fairly easy fashion. Consider random variable

$$x = \sum_{k=1}^{K} r_k^{\nu_k} , \tag{61}$$

where $\{r_k\}$ are arbitrary random variables, statistically independent of each other, and with different distributions. Power ν_k is arbitrary (except that ν_k must be a positive integer for those r_k that can become negative). Let the probability density function of random variable r_k be $p_k(v)$. Then the characteristic function of $r_k^{\nu_k}$ is

$$g_k(\xi) = \overline{\exp\left(i\xi r_k^{\nu_k}\right)} = \int dv \, \exp\left(i\xi v^{\nu_k}\right) p_k(v) =$$

$$= \frac{1}{\nu_k} \int \frac{dt}{t} \exp(i\xi t) \, t^{1/\nu_k} \, p_k\left(t^{1/\nu_k}\right) . \tag{62}$$

If (62) is not integrable in closed form, it can be evaluated by means of an FFT (one for each k if the probability density functions or ν_k are all different). Then the characteristic function of random variable x in (61) is given by

$$f_x(\xi) = \prod_{k=1}^{K} \left\{ g_k(\xi) \right\} . \tag{63}$$

Now the techniques of this report are directly applicable to (63).

An additional example is afforded by

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$$x = \sum_{k=1}^{K} \alpha_k\, v_k^2 + \left(\sum_{k=1}^{K} \beta_k\, v_k\right)^2 + \sum_{k=1}^{K} \gamma_k\, v_k, \qquad (64)$$

where $\{\alpha_k\}$, $\{\beta_k\}$, and $\{\gamma_k\}$ are constants, and $\{v_k\}$ are independent random variables with arbitrary probability density functions. The characteristic function of x is

$$f_x(\xi) = \overline{\exp(i\xi x)} = \int dV\, p_v(V)\, \exp\left(i\xi\left[\sum \alpha_k v_k^2 + \left(\sum \beta_k v_k\right)^2 + \sum \gamma_k v_k\right]\right), \qquad (65)$$

where $V = (v_1, v_2, \ldots, v_K)$. Now since

$$\left(\frac{ia}{\pi}\right)^{1/2}\int dy\, \exp(-iay^2 + iby) = \exp\left(\frac{ib^2}{4a}\right) \quad \text{for } a \neq 0, \qquad (66)$$

we identify $a = \xi/4$, $b = \xi\sum \beta_k v_k$, eliminate the square in the exponent, and express (65) as

$$f_x(\xi) = \int dV\, p_v(V)\, \exp\left(i\xi\sum \alpha_k v_k^2 + i\xi\sum \gamma_k v_k\right) \cdot$$

$$\cdot \left(\frac{i\xi}{4\pi}\right)^{1/2}\int dy\, \exp\left(-\frac{i\xi y^2}{4} + i\xi y\sum \beta_k v_k\right) =$$

$$= \left(\frac{i\xi}{4\pi}\right)^{1/2}\int dy\, \exp\left(-\frac{i\xi y^2}{4}\right)\prod_{k=1}^{K}\left\{\int dv_k\, p_k(v_k)\, \exp(i\xi(\alpha_k v_k^2 + \gamma_k v_k + y\beta_k v_k))\right\}, \quad (67)$$

where

$$p_v(V) = \prod_{k=1}^{K}\left\{p_k(v_k)\right\}. \qquad (68)$$

The inner integrals in (67) can either be done analytically or numerically. Then the remaining single integral on y must be numerically evaluated to find characteristic function $f_x(\xi)$. As an example, if v_k is exponentially distributed

$$p_k(v) = a_k \exp(-a_k v) \text{ for } v > 0, \tag{69}$$

then the inner integrals in (67) are w-functions; see ref. 13, ch. 7. A simpler method of handling general quadratic expressions like (64) with Gaussian V is presented in ref. 15.

SUMMARY

An accurate method for efficient evaluation of the cumulative and exceedance distribution functions has been derived and applied to several examples to illustrate its utility. Choice of the sampling increment Δ applied to the characteristic function controls the aliasing problem, and selection of the limit L minimizes the tail error; the effects of both of these parameters can be observed from sample plots of the distributions and can be modified if needed. Additionally, shift b must be chosen so as to yield a positive random variable with probability virtually 1. The number of distribution values yielded depends on the size of the FFT employed and can be independently selected to yield the desired spacing in distribution values.

APPENDIX A. SAMPLING FOR A FOURIER TRANSFORM

Suppose we are interested in evaluating Fourier transform

$$G(f) = \int dt \, \exp(-i2\pi ft) \, g(t). \tag{A-1}$$

If we sample at interval Δ in t in (A-1), and use integration weighting $w(t)$, we have the approximation to $G(f)$,

$$\tilde{G}(f) \equiv \int dt \, \exp(-i2\pi ft) \, g(t) \, \delta_\Delta(t) \, w(t)$$

$$= G(f) \circledast \frac{1}{\Delta} \delta_{\frac{1}{\Delta}}(f) \circledast W(f)$$

$$= \frac{1}{\Delta} \sum_n G(f - \frac{n}{\Delta}) \circledast W(f), \tag{A-2}$$

where infinite impulse train (sampling function)

$$\delta_\Delta(t) = \sum_n \delta(t - n\Delta), \tag{A-3}$$

and $\circledast$ denotes convolution.

The term

$$\frac{1}{\Delta} \sum_n G(f - \frac{n}{\Delta}) \tag{A-4}$$

in (A-2) is an infinitely aliased version of desired function $G(f)$; this aliasing is an unavoidable effect due to sampling at increment Δ. However, to minimize any further aliasing in (A-2), we would like $W(f) = \delta(f)$, which requires $w(t) = 1$ for all t; strictly, all we need is

$$w(n\Delta) = 1 \quad \text{for all } n. \tag{A-5}$$

That is, the best weighting in (A-2) is uniform.

As an example, for Simpson's rule, we have weighting

$$w(n\Delta) = \ldots, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \ldots = 1 + \frac{1}{3}(-1)^n \text{ or } 1 - \frac{1}{3}(-1)^n, \qquad \text{(A-6)}$$

which can be represented as samples of time function

$$w(t) = 1 + \frac{1}{3}\exp(i\pi t/\Delta) \text{ or } 1 - \frac{1}{3}\exp(i\pi t/\Delta). \qquad \text{(A-7)}$$

The corresponding transform is

$$W(f) = \int dt \exp(-i2\pi ft) \, w(t) =$$

$$= \delta(f) + \frac{1}{3}\delta(f - \frac{1}{2\Delta}) \quad \text{or} \quad \delta(f) - \frac{1}{3}\delta(f - \frac{1}{2\Delta}). \qquad \text{(A-8)}$$

But this window function substituted in (A-2) results in an extra aliasing lobe in $\widetilde{G}(f)$, halfway between the unavoidable major lobes of (A-4) at multiples of $1/\Delta$, of magnitude $1/3$ as large. This effect very adversely affects the quality of $\widetilde{G}(f)$ insofar as its approximation to the desired $G(f)$ is concerned. Thus the best sampling plan in (A-2) is the equal weight structure of (A-5) when one wants to approximate the Fourier transform of (A-1). For a bounded region, this is modified to the Trapezoidal rule, i.e., half-size weights at the boundaries.

APPENDIX B.   LISTINGS OF PROGRAMS FOR FIVE EXAMPLES

The following listings are programs in BASIC for the Hewlett Packard 9845B Desktop Calculator. The FFT utilized is one with the capability of a zero subscript and is listed at the end of the appendix. Mathematically, the FFT programmed is

$$Z_m = \sum_{k=0}^{M-1} \exp(-i2\pi mk/M) \, z_k \quad \text{for } 0 \le m \le M-1,$$

where the arrays $\{z_k\}_0^{M-1}$ and $\{Z_m\}_0^{M-1}$ are handled directly, including the zero-subscript terms z_0 and Z_0.

A detailed explanation of the first program below for Chi-Squared random variables is as follows: line 20 specifies the parameter K, where 2K is the number of squared-Gaussian random variables summed to yield random variable x. Lines 30-60 require inputs L, Δ, b, M respectively, on the part of the user. Line 110 is the input of mean μ_x of random variable x, as evaluated analytically from characteristic function $f_x(\xi)$. Lines 180-210 specifically evaluate the characteristic function $f_y(\xi)$ at general point ξ. All of these lines mentioned thus far require inputs on the part of the user and are so noted in the listing by the presence of a single ! on each line; the comments after a double !! are for information purposes only and need not be modified. This convention is also adopted in the remaining listings.

Lines 220-240 accomplish the collapsing operation of (32)-(33). The cumulative and exceedance distribution functions are finally evaluated and stored in arrays X(*) and Y(*) in lines 400-410.

Some further elaboration is necessary for the listing of the Smirnov characteristic function as given by (42). Since a characteristic function is a continuous function of real ξ, the square root in (42) is not a principal value square root, but in fact must yield a continuous function in ξ. In

order to achive this, the argument of the square root is traced continuously from $\xi=0$ (line 110). If an abrupt change in phase is detected, a polarity indicator takes note of this fact (line 250) and corrects the final values of characteristic function $f_y(\xi)$ (lines 260-270). No problems are encountered with complex $\sin(z)$ since it is analytic for all z.

```
10  ! CHI-SQUARE CHARACTERISTIC FUNCTION 1/(1-i 2 xi)^4
20      K=4                      !  2K=8 degrees of freedom
30      L=200                    !  Limit on integral of char. function
40      Delta=.075               !  Sampling increment on char. function
50      Bs=0                     !  Shift b
60      M=2^8                    !  Size of FFT
70      PRINTER IS 0
80      PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
90      REDIM X(0:M-1),Y(0:M-1)
100     DIM X(0:1023),Y(0:1023)
110     Mux=2*K                               !  Mean of random variable x
123     Muy=Mux+Bs
130     X(0)=0
140     Y(0)=.5*Delta*Muy
150     N=INT(L/Delta)
160     FOR Ns=1 TO N
170     Xi=Delta*Ns                           !!  Argument xi of char. fn.
180     C=Xi+Xi                               !  Calculation of
190     CALL Mul(1,-C,1,-C,A,B)               !  characteristic
200     CALL Mul(A,B,A,B,C,D)                 !  function fy(xi)
210     CALL Div(1,0,C,D,Fyr,Fyi)             !  for K=4
220     Ms=Ns MOD M                           !!  Collapsing
230     X(Ms)=X(Ms)+Fyr/Ns
240     Y(Ms)=Y(Ms)+Fyi/Ns
250     NEXT Ns
260     CALL Fft10z(M,X(*),Y(*))              !!  0 subscript FFT
```

```
270     PLOTTER IS "GRAPHICS"
280     GRAPHICS
290     SCALE 0,M,-14,0
300     LINE TYPE 3
310     GRID M/8,1
320     PENUP
330     LINE TYPE 1
340     B=Bs*M*Delta/(2*PI)              !!  Origin for random variable x
350     MOVE B,0
360     DRAW B,-14
370     PENUP
380     FOR Ks=0 TO M-1
390     T=Y(Ks)/PI-Ks/M
400     X(Ks)=.5-T                       !!  Cumulative probability in X(*)
410     Y(Ks)=Pr=.5+T                    !!  Exceedance probability in Y(*)
420     IF Pr>=1E-12 THEN Y=LGT(Pr)
430     IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
440     IF ABS(Pr)<1E-12 THEN Y=-12
450     PLOT Ks,Y
460     NEXT Ks
470     PENUP
480     PRINT Y(0);Y(1);Y(M-2);Y(M-1)         .
490     FOR Ks=0 TO M-1
509     Pr=X(Ks)
510     IF Pr>=1E-12 THEN Y=LGT(Pr)
520     IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
530     IF ABS(Pr)<1E-12 THEN Y=-12
540     PLOT Ks,Y
550     NEXT Ks
560     PENUP
570     PAUSE
580     DUMP GRAPHICS
590     PRINT LIN(5)
600     PRINTER IS 16
610     END
620     !
630     SUB Mul(X1,Y1,X2,Y2,A,B)         !   Z1*Z2
640     A=X1*X2-Y1*Y2
650     B=X1*Y2+X2*Y1
660     SUBEND
670     !
680     SUB Div(X1,Y1,X2,Y2,A,B)         !   Z1/Z2
690     T=X2*X2+Y2*Y2
700     A=(X1*X2+Y1*Y2)/T
710     B=(Y1*X2-X1*Y2)/T
720     SUBEND
730     !
740     SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER    0 subscript
```

```
10  ! GAUSSIAN CHARACTERISTIC FUNCTION exp(-.5 xi^2)
20      L=7                         !  Limit on integral of char. function
30      Delta=.3                    !  Sampling increment on char. function
40      Bs=.375*(2*PI/Delta) !  Shift b, as fraction of alias interval
50      M=2^8                       !  Size of FFT
60      PRINTER IS 0
70      PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
80      REDIM X(0:M-1),Y(0:M-1)
90      DIM X(0:1023),Y(0:1023)
100     Mux=0                                       !  Mean of random variable x
110     Muy=Mux+Bs
120     X(0)=0
130     Y(0)=.5*Delta*Muy
140     N=INT(L/Delta)
150     FOR Ns=1 TO N
160     Xi=Delta*Ns                     !!  Argument xi of char. fn.
170     A=EXP(-.5*Xi*Xi)                !   Calculation of
180     B=Bs*Xi                         !   characteristic
190     Fyr=A*COS(B)                    !   function
200     Fyi=A*SIN(B)                    !   fy(xi)
210     Ms=Ns MOD M                     !!  Collapsing
220     X(Ms)=X(Ms)+Fyr/Ns
230     Y(Ms)=Y(Ms)+Fyi/Ns
240     NEXT Ns
250     CALL Fft10z(M,X(*),Y(*))        !!  0 subscript FFT
260     PLOTTER IS "GRAPHICS"
270     GRAPHICS
280     SCALE 0,M,-14,0
290     LINE TYPE 3
300     GRID M/8,1
310     PENUP
320     LINE TYPE 1
330     B=Bs*M*Delta/(2*PI)             !!  Origin for random variable x
340     MOVE B,0
350     DRAW B,-14
360     PENUP
370     FOR Ks=0 TO M-1
380     T=Y(Ks)/PI-Ks/M
390     X(Ks)=.5-T                      !!  Cumulative probability in X(*)
400     Y(Ks)=Pr=.5+T                   !!  Exceedance probability in Y(*)
410     IF Pr>=1E-12 THEN Y=LGT(Pr)
420     IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
430     IF ABS(Pr)<1E-12 THEN Y=-12
440     PLOT Ks,Y
450     NEXT Ks
460     PENUP
470     PRINT Y(0);Y(1);Y(M-2);Y(M-1)
480     FOR Ks=0 TO M-1
490     Pr=X(Ks)
500     IF Pr>=1E-12 THEN Y=LGT(Pr)
510     IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
520     IF ABS(Pr)<1E-12 THEN Y=-12
530     PLOT Ks,Y
540     NEXT Ks
550     PENUP
560     PAUSE
570     DUMP GRAPHICS
580     PRINT LIN(5)
590     PRINTER IS 16
600     END
610     !
620     SUB Fft10z(N,X(*),Y(*))      ! N <= 2^10 = 1024, N=2^INTEGER    0 subscript
```

B-4

```
10  ! SMIRNOV CHARACTERISTIC FUNCTION [s/sin(s)]^1/2 where s=(1+i)sqr(xi)
20     L=3000                    !  Limit on integral of char. function
30     Delta=1                   !  Sampling increment on char. function
40     Bs=0                      !  Shift b
50     M=2^8                     !  Size of FFT
60     PRINTER IS 0
70     PRINT "L =";L,"Delta =";Delta,"b =",Bs,"M =";M
80     REDIM X(0:M-1),Y(0:M-1)
90     DIM X(0:1023),Y(0:1023)
100    Mux=1/6                             !  Mean of random variable x
110    R=0                                 !  Argument of square root
120    P=1                                 !  Polarity indicator
130    Muy=Mux+Bs
140    X(0)=0
150    Y(0)=.5*Delta*Muy
160    N=INT(L/Delta)
170    FOR Ns=1 TO N
180    Xi=Delta*Ns                         !! Argument xi of char. fn.
190    A=SQR(Xi)                           !  Calculation
200    CALL Sin(A,A,B,C)                   !  of
210    CALL Div(A,A,B,C,D,E)               :  characteristic
220    CALL Sqr(D,E,A,B)                   !  function
230    Ro=R                                !  fy(xi)
240    R=ATN(B/A)                          !
250    IF ABS(R-Ro)>1.6 THEN P=-P          !
260    Fyr=A*P                             !
270    Fyi=B*P                             !
280    Ms=Ns MOD M                         !! Collapsing
290    X(Ms)=X(Ms)+Fyr/Ns
300    Y(Ms)=Y(Ms)+Fyi/Ns
310    NEXT Ns
320    CALL Fft10z(M,X(*),Y(*))            !! 0 subscript FFT
330    PLOTTER IS "GRAPHICS"
340    GRAPHICS
350    SCALE 0,M,-14,0
360    LINE TYPE 3
370    GRID M/8,1
380    PENUP
390    LINE TYPE 1
400    B=Bs*M*Delta/(2*PI)                 !! Origin for random variable x
410    MOVE B,0
420    DRAW B,-14
430    PENUP
440    FOR Ks=0 TO M-1
450    T=Y(Ks)/PI-Ks/M
460    X(Ks)=.5-T                          !! Cumulative probability in X(*)
470    Y(Ks)=Pr=.5+T                       !! Exceedance probability in Y(*)
480    IF Pr>=1E-12 THEN Y=LGT(Pr)
490    IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
500    IF ABS(Pr)<1E-12 THEN Y=-12
510    PLOT Ks,Y
520    NEXT Ks
```

```
530     PENUP
540     PRINT Y(0);Y(1);Y(M-2);Y(M-1)
550     FOR Ks=0 TO M-1
560     Pr=X(Ks)
570     IF Pr>=1E-12 THEN Y=LGT(Pr)
580     IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
590     IF ABS(Pr)<1E-12 THEN Y=-12
600     PLOT Ks,Y
610     NEXT Ks
620     PENUP
630     PAUSE
640     DUMP GRAPHICS
650     PRINT LIN(5)
660     PRINTER IS 16
670     END
680     !
690     SUB Div(X1,Y1,X2,Y2,A,B)          !  Z1/Z2
700     T=X2*X2+Y2*Y2
710     A=(X1*X2+Y1*Y2)/T
720     B=(Y1*X2-X1*Y2)/T
730     SUBEND
740     !
750     SUB Sqr(X,Y,A,B)                  !  PRINCIPAL SQR(Z)
760     IF X<>0 THEN 800
770     A=B=SQR(.5*ABS(Y))
780     IF Y<0 THEN B=-B
790     GOTO 910
800     F=SQR(SQR(X*X+Y*Y))
810     T=.5*ATN(Y/X)
820     A=F*COS(T)
830     B=F*SIN(T)
840     IF X>0 THEN 910
850     T=A
860     A=-B
870     B=T
880     IF Y>=0 THEN 910
890     A=-A
900     B=-B
910     SUBEND
920     !
930     SUB Sin(X,Y,A,B)                  !  SIN(Z)
940     E=EXP(Y)
950     A=.5*SIN(X)*(E+1/E)
960     IF ABS(Y)<.1 THEN 990
970     S=.5*(E-1/E)
980     GOTO 1010
990     S=Y*Y
1000    S=Y*(120+S*(20+S))/120
1010    B=COS(X)*S
1020    SUBEND
1030    !
1040    SUB Fft10z(N,X(*),Y(*))      ! N <= 2^10 = 1024, N=2^INTEGER    0 subscript
```

```
10  ! NON-CENTRAL CHI-SQUARE CHARACTERISTIC FUNCTION
20  ! exp(1 d^2 xi / s) / s^nu    where s = 1-1 2 xi
30      Nu=2.7                      ! Power law nu
40      Ds=3                        ! Deflection d
50      L=500                       ! Limit on integral of char. function
60      Delta=.05                   ! Sampling increment on char. function
70      Bs=0                        ! Shift b
80      M=2^8                       ! Size of FFT
90      PRINTER IS 0
100     PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
110     REDIM X(0:M-1),Y(0:M-1)
120     DIM X(0:1023),Y(0:1023)
130     D2=Ds*Ds                                  ! Calculate parameter
140     Mux=2*Nu+D2                               ! Mean of random variable x
150     Muy=Mux+Bs
160     X(0)=0
170     Y(0)=.5*Delta*Muy
180     N=INT(L/Delta)
190     FOR Ns=1 TO N
200     Xi=Delta*Ns                               !! Argument xi of char. fn.
210     T=Xi+Xi                                   ! Calculation of
220     CALL Div(0,D2*Xi,1,-T,A,B)                ! characteristic
230     CALL Log(1,-T,C,D)                        ! function
240     CALL Exp(A-Nu*C,B-Nu*D+Bs*Xi,Fyr,Fyi)    ! fy(xi)
250     Ms=Ns MOD M                               !! Collapsing
260     X(Ms)=X(Ms)+Fyr/Ns
270     Y(Ms)=Y(Ms)+Fyi/Ns
280     NEXT Ns
290     CALL Fft10z(M,X(*),Y(*))                  !! 0 subscript FFT
300     PLOTTER IS "GRAPHICS"
310     GRAPHICS
320     SCALE 0,M,-14,0
330     LINE TYPE 3
340     GRID M/8,1
350     PENUP
360     LINE TYPE 1
370     B=Bs*M*Delta/(2*PI)                       !! Origin for random variable x
380     MOVE B,0
390     DRAW B,-14
400     PENUP
410     FOR Ks=0 TO M-1
420     T=Y(Ks)/PI-Ks/M
430     X(Ks)=.5-T                                !! Cumulative probability in X(*)
440     Y(Ks)=Pr=.5+T                             !! Exceedance probability in Y(*)
450     IF Pr>=1E-12 THEN Y=LGT(Pr)
460     IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
470     IF ABS(Pr)<1E-12 THEN Y=-12
480     PLOT Ks,Y
490     NEXT Ks
500     PENUP
```

```
510     PRINT Y(0);Y(1);Y(M-2);Y(M-1)
520     FOR Ks=0 TO M-1
530     Pr=X(Ks)
540     IF Pr>=1E-12 THEN Y=LGT(Pr)
550     IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
560     IF ABS(Pr)<1E-12 THEN Y=-12
570     PLOT Ks,Y
580     NEXT Ks
590     PENUP
600     PAUSE
610     DUMP GRAPHICS
620     PRINT LIN(5)
630     PRINTER IS 16
640     END
650     !
660     SUB Div(X1,Y1,X2,Y2,A,B)          !   Z1/Z2
670     T=X2*X2+Y2*Y2
680     A=(X1*X2+Y1*Y2)/T
690     B=(Y1*X2-X1*Y2)/T
700     SUBEND
710     !
720     SUB Exp(X,Y,A,B)                  !   EXP(Z)
730     T=EXP(X)
740     A=T*COS(Y)
750     B=T*SIN(Y)
760     SUBEND
770     !
780     SUB Log(X,Y,A,B)                  !   PRINCIPAL LOG(Z)
790     A=.5*LOG(X*X+Y*Y)
800     IF X<>0 THEN 830
810     B=.5*PI*SGN(Y)
820     GOTO 850
830     B=ATN(Y/X)
840     IF X<0 THEN B=B+PI*(1-2*(Y<0))
850     SUBEND
860     !
870     SUB Fft10z(N,X(*),Y(*))    ! N <= 2^10 = 1024, N=2^INTEGER      0 subscript
```

```
10 ! GAUSSIAN PRODUCT CHARACTERISTIC FUNCTION (56)
20     Nu=7.7                   ! Power Nu
30     Rho=-.3                  ! Correlation coefficient
40     L=5                      ! Limit on integral of char. function
50     Delta=.06                ! Sampling increment on char. function
60     Bs=.5*(2*PI/Delta)       ! Shift b, as fraction of alias interval
70     M=2^9                    ! Size of FFT
80     PRINTER IS 0
90     PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
100    REDIM X(0:M-1),Y(0:M-1)
110    DIM X(0:1023),Y(0:1023)
120    T1=1-Rho*Rho                          ! Calculate
130    T2=2*Rho                              ! parameters
140    Mux=2*Nu*Rho                          ! Mean of random variable x
150    Muy=Mux+Bs
160    X(0)=0
170    Y(0)=.5*Delta*Muy
180    N=INT(L/Delta)
190    FOR Ns=1 TO N
200    Xi=Delta*Ns                           !! Argument xi of char. fn.
210    CALL Log(1+T1*Xi*Xi,-T2*Xi,A,B)    ! Calculation of
220    CALL Exp(-Nu*A,Bs*Xi-Nu*B,Fyr,Fyi)! characteristic function fy(xi)
230    Ms=Ns MOD M                           !! Collapsing
240    X(Ms)=X(Ms)+Fyr/Ns
250    Y(Ms)=Y(Ms)+Fyi/Ns
260    NEXT Ns
270    CALL Fft10z(M,X(*),Y(*))              !! 0 subscript FFT
280    PLOTTER IS "GRAPHICS"
290    GRAPHICS
300    SCALE 0,M,-14,0
310    LINE TYPE 3
320    GRID M/8,1
330    PENUP
340    LINE TYPE 1
350    B=Bs*M*Delta/(2*PI)                   !! Origin for random variable x
360    MOVE B,0
370    DRAW B,-14
380    PENUP
```

```
390    FOR Ks=0 TO M-1
400    T=Y(Ks)/PI-Ks/M
410    X(Ks)=.5-T                          !!   Cumulative probability in X(*)
420    Y(Ks)=Pr=.5+T                        !!   Exceedance probability in Y(*)
430    IF Pr>=1E-12 THEN Y=LGT(Pr)
440    IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
450    IF ABS(Pr)<1E-12 THEN Y=-12
460    PLOT Ks,Y
470    NEXT Ks
480    PENUP
490    PRINT Y(0);Y(1);Y(M-2);Y(M-1)
500    FOR Ks=0 TO M-1
510    Pr=X(Ks)
520    IF Pr>=1E-12 THEN Y=LGT(Pr)
530    IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
540    IF ABS(Pr)<1E-12 THEN Y=-12
550    PLOT Ks,Y
560    NEXT Ks
570    PENUP
580    PAUSE
590    DUMP GRAPHICS
600    PRINT LIN(5)
610    PRINTER IS 16
620    END
630    !
640    SUB Exp(X,Y,A,B)                     !   EXP(Z)
650    T=EXP(X)
660    A=T*COS(Y)
670    B=T*SIN(Y)
680    SUBEND
690    !
700    SUB Log(X,Y,A,B)                     !   PRINCIPAL LOG(Z)
710    A=.5*LOG(X*X+Y*Y)
720    IF X<>0 THEN 750
730    B=.5*PI*SGN(Y)
740    GOTO 770
750    B=ATN(Y/X)
760    IF X<0 THEN B=B+PI*(1-2*(Y<0))
770    SUBEND
780    !
790    SUB Fft10z(N,X(*),Y(*))       ! N <= 2^10 = 1024, N=2^INTEGER      0 subscript
```

```
10      SUB Fft10z(N,X(*),Y(*))      ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
20      DIM C(0:256)
30      INTEGER I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,J,K
40      DATA 1,.999981175283,.999924701839,.999830581796,.999698818696,.9995294175
01,.999322384588,.999077727753,.998795456205,.998475580573,.998118112900
50      DATA .997723066644,.997290456679,.996820299291,.996312612183,.995767414468
,.995184726672,.994564570734,.993906970002,.993211949235,.992479534599
60      DATA .991709753669,.990902635428,.990058210262,.989176509965,.988257567731
,.987301418158,.986308097245,.985277642389,.984210092387,.983105487431
70      DATA .981963869110,.980785280403,.979569765685,.978317370720,.977028142658
,.975702130039,.974339382786,.972939952206,.971503890986,.970031253195
80      DATA .968522094274,.966976471045,.965394441698,.963776065795,.962121404269
,.960430519416,.958703474896,.956940335732,.955141168306,.953306040354
90      DATA .951435020969,.949528180593,.947585591018,.945607325381,.943593458162
,.941544065183,.939459223602,.937339011913,.935183509939,.932992798835
100     DATA .930766961079,.928506080473,.926210242138,.923879532511,.921514039342
,.919113851690,.916679059921,.914209755704,.911706032005,.909167983091
110     DATA .906595704515,.903989293123,.901348847046,.898674465694,.895966249756
,.893224301196,.890448723245,.887639620403,.884797098431,.881921264348
120     DATA .879012226429,.876070094195,.873094978418,.870086991109,.867046245516
,.863972856122,.860866938638,.857728610000,.854557988365,.851355193105
130     DATA .848120344803,.844853565250,.841554977437,.838224705555,.834862874986
,.831469612303,.828045045258,.824589302785,.821102514991,.817584813152
140     DATA .814036329706,.810457198253,.806847553544,.803207531481,.799537269108
,.795836904609,.792106577300,.788346427627,.784556597156,.780737228572
150     DATA .776888465573,.773010453363,.769103337645,.765167265622,.761202385484
,.757208846506,.753186799044,.749136394523,.745057785441,.740951125355
160     DATA .736816568877,.732654271672,.728464390448,.724247082951,.720002507961
,.715730825284,.711432195745,.707106781187,.702754744457,.698376249409
170     DATA .693971460890,.689540544737,.685083667773,.680600997795,.676092703575
,.671558954847,.666999922304,.662415777590,.657806693297,.653172842954
180     DATA .648514401022,.643831542890,.639124444864,.634393284164,.629638238915
,.624859488142,.620057211763,.615231590581,.610382806276,.605511041404
190     DATA .600616479384,.595699304492,.590759701859,.585797857456,.580813958096
,.575808191418,.570780745387,.565731810784,.560661576197,.555570233020
200     DATA .550457972937,.545324988422,.540171472730,.534997619887,.529803624686
,.524589682678,.519355990166,.514102744193,.508830142543,.503538383726
210     DATA .498227666973,.492898192230,.487550160148,.482183772079,.476799230063
,.471396736826,.465976495768,.460538710958,.455083587126,.449611329655
220     DATA .444122144570,.438616238539,.433093818853,.427555093430,.422000270800
,.416429560098,.410843171058,.405241314005,.399624199846,.393992040061
230     DATA .388345046699,.382683432365,.377007410216,.371317193952,.365612997805
,.359895036535,.354163525420,.348418680249,.342660717312,.336889853392
240     DATA .331106305760,.325310292162,.319502030816,.313681740399,.307849640042
,.302005949319,.296150888244,.290284677254,.284407537211,.278519689385
250     DATA .272621355450,.266712757475,.260794117915,.254865659605,.248927605746
,.242980179903,.237023605994,.231058108281,.225083911360,.219101240157
260     DATA .213110319916,.207111376192,.201104634842,.195090322016,.189068664150
,.183039887955,.177004220412,.170961888760,.164913120490,.158858143334
270     DATA .152797185258,.146730474455,.140658239333,.134580708507,.128498110794
,.122410675199,.116318630912,.110222207294,.104121633872,.980171403296E-1
280     DATA .919089564971E-1,.857973123444E-1,.796824379714E-1,.735645635997E-1,.
674439195637E-1,.613207363022E-1,.551952443497E-1,.490676743274E-1
290     DATA .429382569349E-1,.368072229414E-1,.306748031766E-1,.245412285229E-1,.
184067299058E-1,.122715382857E-1,.613588464915E-2,0
300     READ C(*)
310     K=1024/N
320     FOR J=0 TO N/4
330     C(J)=C(K*J)
340     NEXT J
350     N1=N/4
360     N2=N1+1
370     N3=N2+1
```

```
380     N4=N1+N3
390     Log2n=INT(1.4427*LOG(N)+.5)
400     FOR I1=1 TO Log2n
410     I2=2^(Log2n-I1)
420     I3=2*I2
430     I4=N/I3
440     FOR I5=1 TO I2
450     I6=(I5-1)*I4+1
460     IF I6<=N2  HEN 500
470     N6=-C(N4-I6-1)
480     N7=-C(I6-N1-1)
490     GOTO 520
500     N6=C(I6-1)
510     N7=-C(N3-I6-1)
520     FOR I7=0 TO N-I3 STEP I3
530     I8=I7+I5
540     I9=I8+I2
550     N8=X(I8-1)-X(I9-1)
560     N9=Y(I8-1)-Y(I9-1)
570     X(I8-1)=X(I8-1)+X(I9-1)
580     Y(I8-1)=Y(I8-1)+Y(I9-1)
590     X(I9-1)=N6*N8-N7*N9
600     Y(I9-1)=N6*N9+N7*N8
610     NEXT I7
620     NEXT I5
630     NEXT I1
640     I1=Log2n+1
650     FOR I2=1 TO 10                    ' 2^10=1024
660     C(I2-1)=1
670     IF I2>Log2n THEN 690
680     C(I2-1)=2^(I1-I2)
690     NEXT I2
700     k=1
710     FOR I1=1 TO C(9)
720     FOR I2=I1 TO C(8) STEP C(9)
730     FOR I3=I2 TO C(7) STEP C(8)
740     FOR I4=I3 TO C(6) STEP C(7)
750     FOR I5=I4 TO C(5) STEP C(6)
760     FOR I6=I5 TO C(4) STEP C(5)
770     FOR I7=I6 TO C(3) STEP C(4)
780     FOR I8=I7 TO C(2) STEP C(3)
790     FOR I9=I8 TO C(1) STEP C(2)
800     FOR I10=I9 TO C(0) STEP C(1)
810     J=I10
820     IF k>J THEN 890
830     A=X(k-1)
840     X(k-1)=X(J-1)
850     X(J-1)=A
860     A=Y(k-1)
870     Y(k-1)=Y(J-1)
880     Y(J-1)=A
890     k=k+1
900     NEXT I10
910     NEXT I9
920     NEXT I8
930     NEXT I7
940     NEXT I6
950     NEXT I5
960     NEXT I4
970     NEXT I3
980     NEXT I2
990     NEXT I1
1000    SUBEND
```

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ONR (ONR-200, -220, -400, -410, -414, -425AC)	6
CNM (MAT-05, (Capt. Z. L. Newcomb), SP-20, ASW-14)	3
NRL	1
NORDA	1
NAVOCEANO (Code 02)	1
NAVAIRSYSCOM, (PMA/PMS-266)	1
NAVELESYSYCOM (ELEX 03, PME-124, ELEX 612)	3
NAVSEASYSCOM (SEA-63R (2), -63D, -63Y, PMS-411, -409)	6
NAVAIRDEVCEN, Warminster	4
NAVAIRDEVCEN, Key West	1
NOSC	4
NOSC, Library (Code 6565)	1
NAVWPNSCEN	2
NCSC	2
NAVCIVENGRLAB	1
NAVSWC	4
NAVSWC, White Oak Lab.	4
DWTNSRDC, Annapolis	2
DWINSRDC, Bethesda	2
NAVTRAEQUIPCENT (Tech. Library)	1
NAVPGSCOL	4
APL/UW, Seattle	1
ARL/Penn State	2
ARL, University of Texas	1
DTIC	2
DARPA (CDR K. Evans)	1
Woods Hole Oceanographic Institution	1
MPL/Scripps	1